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UNRESTRICTED RICCATIAN SOLUTION FIELDS

By A. WINTNER (*Johns Hopkins*)

[Received 12 February 1946]

If x and y in $y' = dy/dx$ are restricted to the real field, then the x-axis represents the only solution path of the differential equation $y' = \frac{1}{2}y^3$ which exists on the whole half-line $0 \leqslant x < \infty$. In fact, all solutions $y = y(x)$ distinct from the envelope $y = 0$ are of the form $y = \pm(x_0 - x)^{-\frac{1}{2}}$, where x_0 is an integration constant. But this solution does not exist for any non-negative x or ceases to exist as $x \to x_0 - 0$ according as x_0 is non-positive or positive.

The following considerations will centre about a criterion which, for a differential equation $y' = f(x,y)$ in which f is subject to certain *qualitative* restrictions, prevents such a situation. In other words, whereas the general existence theorem for the initial-value problem

$$y' = f(x,y), \qquad y(0) = y_0$$

supplies a solution $y = y(x)$ only on a 'small' interval $0 \leqslant x < x^0$ (and, as shown by the above example, it cannot supply more), the qualitative criterion in question will ensure existence in the large. Needless to say, the restrictions will be placed on the *data*, that is, on the function $f(x,y)$ and on the integration constant $y(0)$.

The need for such criteria arose in connexion with Riccati's differential equation,

$$\text{(R)} \qquad y' = a(x)y^2 + b(x)y + c(x),$$

where the coefficients $a(x)$, $b(x)$, $c(x)$ are real-valued, continuous functions of x (the mere continuity of the coefficients is sufficient, even for uniqueness, since (R) satisfies Lipschitz's condition at every point of the (x,y)-plane). What can reasonably be expected here is revealed by the simplest case of (R), namely, by the case of coefficients $a\ (\neq 0)$, b, c which are independent of x. For instance, if $a = -1$ and $b = 0$, $c = 0$, then, since (R) becomes $y' = -y^2$, all solutions $y = y(x)$ distinct from their common envelope, $y = 0$, are of the form $y = (x - x_0)^{-1}$, where $x_0 = -1/y(0)$. Hence, every solution $y(x)$ assigned by a negative initial value $y(0)$ will cease to exist at the upper end of a bounded, half-open x-interval $0 \leqslant x < x_0$, whereas the solution exists for $0 \leqslant x < \infty$ if the initial value $y(0)$ is positive

or zero. Thus it is clear that the following theorem (along with its straightforward extensions to cases of a vanishing discriminant d) contains about everything that can reasonably be expected for the non-local existence of the solutions of (R):

(I) *Suppose that the coefficients* $a(x)$, $b(x)$, $c(x)$ *of Riccati's equation* (R) *are real-valued, continuous functions on the half-line* $0 \leqslant x < \infty$ *and have the property that, for every fixed* $x \geqslant 0$, *the roots*

$$y_{\pm}(x) = \tfrac{1}{2}\{-b(x) \pm d(x)^{\frac{1}{2}}\}/a(x) \quad (d = b^2 - 4ac)$$

of the associated quadratic equation are finite, real, and distinct; that is, $a(x) \neq 0$ *and* $d(x) > 0$. *Since* a, b, c, d *go over into* $-a$, b, $-c$, d, *respectively, if* y *is replaced by* $-y$, *it can be assumed that* $a(x) < 0$. *Suppose finally that, at every* $x > 0$, *the value of the smaller root* $y_-(x)$ *is less than or equal to its value* $y_-(0)$ *at* $x = 0$, *while the value of the larger root* $y_+(x)$ *is not less than* $y_+(0)$. *Then all those solutions* $y = y(x)$ *of* (R) *which belong to initial values* $y(0)$ *satisfying the unilateral limitation* $y(0) \geqslant y_-(0)$ *are solutions which exist for* $0 \leqslant x < \infty$.

The method of proof will be such as to supply, under easy conditions, a result concerning the ultimate behaviour of all solutions $y = y(x)$ admitted in (I). What then results is that, no matter how large the initial value $y(0)$ may be, the solution-path $y = y(x)$ is driven under the universal curve $y = y_+(x)$, defined by the last formula-line, whereas it cannot cross the lower universal curve $y = y_-(x)$. And, ultimately, all these solution-paths are monotone increasing.

(II) *Suppose that the coefficients of* (R) *satisfy the assumptions of* (I) *and are, in addition, such as to make the greater root,* $y_+(x)$, *of the associated quadratic equation a non-decreasing function of* x *which does not remain bounded as* $x \to \infty$. *Then all solution-paths* $y = y(x)$ *admitted in* (I) *are ultimately such as to run between the two curves of zero velocity* $y = y_-(x)$ *and* $y = y_+(x)$, *although, ultimately, all these solution-paths are monotone (ascending).*

It is understood that the proviso of 'ultimate' behaviour refers to the exclusion of an x-interval $0 \leqslant x \leqslant x^*$ in which x^* is a function of the integration constant $y(0)$. Trivial examples show that, in general, $x^* \to \infty$ as $y(0) \to \infty$.

An information still more precise, though more superficial, than that supplied by (II) is contained in the following theorem:

(III) *Suppose that the coefficients of* (R) *satisfy the assumptions of*

(II) *and are, in addition, such that there exist two positive numbers L and M satisfying*

$$-\infty < \tfrac{1}{2}b(x)/a(x) < L \quad \textit{if} \quad x \geqslant M.$$

Consider only those solution-paths $y = y(x)$ of (R) *which, besides the unilateral initial restriction $y(0) \geqslant y_-(0)$ of* (I), *are subject to the additional restriction $y(M) > L$. Then, if $y = y_1(x)$ and $y = y_2(x)$ is any pair of such solution-paths of* (R), *the difference $y_1(x)-y_2(x)$ tends to a finite limit as $x \to \infty$.*

Corresponding to the decreasing degree of generality, (III) is the easiest, and (I) the deepest, of these theorems (provided that 'depth' is not measured in terms of the length of the proof). What will make the proof of (I) comparatively deep will be its dependence on a general theorem concerning unspecified differential equations $y' = f(x,y)$ which, though simple enough, was formulated only recently† and which, characteristically, cannot be transferred from the case of a single equation $dy/dx = f(x,y)$ to the case of a system

$$dy_i/dx = f_i(x, y_1,..., y_n) \quad (i = 1,..., n)$$

of more than $n = 1$ equations; cf. for $n = 2$, Painlevé's counter-example, mentioned loc. cit., p. 175 (however, $y_1^2+...+y_n^2 \to \infty$ must hold in the case $n > 1$ also; cf. loc. cit., p. 177).

An easier approach to (I), (II), (III) seems to be furnished by the relevant theorem concerning the separation of the zeros of the solution $u = u(x)$ of a linear differential equation

$$u''+p(x)u'+q(x)u = 0 \tag{r}$$

with continuous coefficients p, q on the one hand, and the connexion between (r) and (R) on the other hand. In fact, if

$$y = -(\log u)'/a(x),$$

the non-linear equation (R) becomes the 'resolvent' of the linear equation (r) where

$$p = -b-a'/a, \quad q = ac.$$

Actually, this formal connexion fails (at least in this form) since the derivative $a' = a'(x)$ need not exist in (R).

What is more important: the method to be applied has nothing to do with the explicit structure of Riccati's equation and leads, therefore, to more general theorems, which cannot be connected at all

† A. Wintner, The infinities in the non-local existence problem of ordinary differential equations, *American J. of Math.* 68 (1946), 173–8.

with linear differential equations. These are theorems (i), (ii), (iii) below. And the above theorems, (I), (II), (III), are the simplest possible illustrations of (i), (ii), (iii) respectively.

Incidentally, a deduction of the properties (I), (II), (III) of (R) from properties of (r) would by no means be straightforward, not even under the necessary assumption of differentiability. Thus (I), (II), (III) should be thought of as facts supplied for (r) by the mapping (R) $\rightarrow$ (r), rather than something to be approached by the inverse mapping (r) $\rightarrow$ (R).

The first of the three general theorems is as follows:

(i) *On the half-plane $x \geqslant 0$, let $f(x,y)$ be a real-valued, continuous function which changes its sign exactly twice on every line $-\infty < y < \infty$, $x = x$, say at*

$$y = y^-(x) \quad \textit{and} \quad y = y^+(x), \quad \textit{where} \quad y^-(x) < y^+(x) \quad (0 \leqslant x < \infty),$$

and† *that these are the only roots of the equation $f(x,y) = 0$. Suppose further that the notation is so chosen as to make the function $f(x,y)$ positive between its zeros (otherwise let y be replaced by $-y$) and that*

$$y^-(0) \geqslant y^-(x) \quad \textit{and} \quad y^+(0) \leqslant y^+(x) \quad (0 \leqslant x < \infty).$$

Then every solution $y = y(x)$ defined, for small x, by any initial value $y(0)$ subject to the lower limitation $y(0) > y^-(0)$ is a solution which exists on the whole half-line $0 \leqslant x < \infty$.

If the initial value $y(0)$ is chosen to be $y^-(0)$, it will follow from the proof of (i) that there exists at least one solution $y = y(x)$ which satisfies this initial condition and exists for $0 \leqslant x < \infty$. Hence, if the (local) solution $y = y(x)$ of the differential equation $y' = f(x,y)$ is unique through every point of the half-plane $x \geqslant 0$ (for instance, if $f(x,y)$ satisfies Lipschitz's condition near every point), then the limitation '$y(0) > y^-(0)$' of (i) may be relaxed to '$y(0) \geqslant y^-(0)$'.

By assumption, $y = y^-(x)$ and $y = y^+(x)$ are continuous curves, say A and B, which do not intersect and reach from $x = 0$ to $x = \infty$. Furthermore, if S denotes that portion of the half-plane $x \geqslant 0$ which is contained between A and B, the function $f(x,y)$ is positive or negative according as the point (x,y) is in S or is not in $S+A+B$. The conditions $y^-(0) \geqslant y^-(x)$ and $y^+(0) \leqslant y^+(x)$ mean that S contains the open strip $x > 0$, $y^-(0) < y < y^+(0)$. Let U and V denote those portions of the half-plane $x \geqslant 0$ which are situated below A and

† This additional restriction is made only for the sake of simplicity.

above B, respectively. Finally, let α denote the point of A on the y-axis, and β the corresponding point of B.

If π is an arbitrary point of the y-axis, there issues from it at least one solution-path $y = y_\pi(x)$ of $y' = f(x,y)$. But this solution-path will, in general, exist only on a 'small' x-interval, say $0 \leqslant x < x_\pi$. And the assertion is that any such solution cannot cease to exist (that is, that $x_\pi = \infty$) if π is chosen above the point α.

Whether π is or is not above α, a solution $y = y_\pi(x)$, if it exists on an interval $0 \leqslant x < x_0$, cannot cease to exist at $x = x_0 = x_0(\pi)$ unless

$$(*) \qquad |y_\pi(x)| \to \infty \quad \text{as} \quad x \to x_0 - 0.$$

As shown (loc. cit.), this is a general theorem concerning a single differential equation $y' = f(x,y)$, in which $f(x,y)$ is continuous on the half-plane $x \geqslant 0$. Hence, it will be sufficient to show that, by virtue of assumptions imposed on $f(x,y)$, there cannot exist an x_0 satisfying (*), if π is chosen above α.

Suppose first that π is chosen on S, that is, between α and β. Then, since $f(x,y)$ is positive, zero, or negative according as (x,y) is on S, $A+B$, or $U+V$, it is seen from $y' = f(x,y)$ that any solution-path $y = y_\pi(x)$ is ascending when x is small. But α is supposed to be the highest point of the curve A. Hence, it is clear from the sign of the derivative $y' = f(x,y)$ that, if a solution $y = y_\pi(x)$ exists on an interval $0 \leqslant x < x_0$, and if x_0 has the property (*), then *ultimately* (that is, when x is close enough to x_0) the solution cannot be in U, the region below A, or even in the region $y \leqslant y^-(0)$, unless it has spent some time in V, the region above B. However, it cannot descend from V into U without passing through S, hence, through the strip $y^-(0) < y < y^+(0)$. But it is ascending while it is in B. Consequently, (*) must take place in V. Thus (*) can be refined to

$$y_\pi(x) \to \infty \quad \text{as} \quad x \to x_0 - 0.$$

On the other hand, in view of $y' = f(x,y)$, the solution $y = y(x)$ ought to be descending, since it is in V. This contradiction proves that (*) cannot take place at all.

Thus far, the initial point π was chosen on the open segment ending at α and β. To extend the proof, let π be either the point β or a point above β. A solution $y = y_\pi(x)$ determined by π cannot reach U or even the region $y \leqslant y^-(0)$, for it would have to pass

through the strip $y^-(0) < y < y^+(0)$, where it is ascending. The proof may now be completed as above; for, if (*) occurs, it must take place in V.

This completes the proof of (i). And (i) implies (I) in virtue of the remark following (I).

The corresponding generalization of (II) is as follows:

(ii) *Suppose that the continuous function $f(x, y)$ satisfies the assumptions of* (i) *and that, in addition, the greater root $y = y^+(x)$ of the equation $f(x, y) = 0$ is a non-decreasing, unbounded function of x. Then, if $y = y^-(x)$ denotes the smaller root of the equation $f(x, y) = 0$, all those solutions $y = y(x)$ of the differential equation $y' = f(x, y)$ which are defined by initial values $y(0)$ satisfying the lower limitation $y(0) > y^-(0)$ represent solution-paths which ultimately—that is, at every point x of a half-line $x^0 < x < \infty$, where x^0 depends on the integration constant $y(0)$—become confined to the fixed belt $y^-(0) < y \leqslant y^+(x)$.*

Every such solution $y = y(x)$ is ultimately non-decreasing.

The last assertion is a corollary of the first, according to which the solution-path must ultimately be in the region $y^-(0) < y \leqslant y^+(x)$. In fact, $f(x, y)$ is non-negative in this region, and so, since $y' = f(x, y)$, a solution-path cannot have any point of descent in this region.

A glance at the proof of (i) shows that what is substantially new in (ii) is the assertion that, no matter how high the initial point π be chosen, a corresponding solution path $y = y_\pi(x)$ will ultimately be depressed to the level of B, and then into S; and that, ultimately, it will be unable to leave $S+B$, or even $y^-(0) < y \leqslant y^+(x)$.

That $y_\pi(x) > y^-(0)$ is satisfied for $0 \leqslant x < \infty$ whenever π is chosen above α was actually shown in the proof of (i).

Next, a solution-path $y = y_\pi(x)$ cannot *stay* in $V+B$. In fact, if it is in $V+B$ for $x_0 \leqslant x < \infty$, its derivative $y' = f(x, y)$ is non-positive, and so $y = y_\pi(x)$ is non-increasing for $x \geqslant x_0$. But then $y_\pi(x) < y^+(x)$: that is, $y = y_\pi(x)$ is in S, for all sufficiently large x, since $y = y^+(x)$ is non-decreasing and unbounded.

Accordingly, any path an arc of which is in V must pass into S. But then it cannot return into V. In fact, the solution-path is descending or ascending according as it is in V or in S, whereas the boundary B, on which y' is 0, is a non-descending curve.

This completes the proof of (II).

It can be expected that, under comparatively light additional restrictions to be placed on $f(x, y)$, all the solution-paths under

consideration will, in some sense, be asymptotic to B, the (upper) curve of zero velocity.

However, caution is necessary, since B itself need not have an asymptotic direction and, if it has one, it need not have an asymptote, since it may become asymptotically perpendicular to the x-axis. In addition, all that is assumed about B is that B is continuous, non-descending, and unbounded; its given parametrization $y = y^+(x)$ need not even be absolutely continuous.

All these possibilities are easily compatible with the assumptions of the following theorem:

(iii) *Let $f(x,y)$ be a continuous function satisfying the assumptions of* (i) *and* (ii) *and having, in addition, the following property: for every sufficiently large fixed x, say for $x \geqslant M$, the function $f(x,y)$ of y alone is a non-increasing function on some fixed half-line $c \leqslant y < \infty$. Then all those solution-paths $y = y(x)$ of the differential equation $y' = f(x,y)$ which are defined by initial values $y(M)$ exceeding* c *(hence, the value of the smaller root of the equation $f(M,y) = 0$) are asymptotically parallel curves; in the sense that, if $y = y_1(x)$ and $y = y_2(x)$ denote any two of these solutions, then, as $x \to \infty$, the difference $y_1(x) - y_2(x)$ tends to a finite limit.*

In the simplest examples illustrating (iii), this limit is 0, and hence independent of $y_1(0)$ and $y_2(0)$. It remains undecided whether this must be the case if nothing is added to the assumptions of (iii).

It can be supposed that $y_1(x) \leqslant y_2(x)$ for every x (for, if these solution-paths cross, the subscripts may be interchanged at the corresponding values of x). Then, if $x \geqslant M$, it follows from the additional assumption of (iii) that $f\{x, y_2(x)\}$ does not exceed $f\{x, y_1(x)\}$. Since $y' = f(x,y)$ is satisfied by $y = y_1$ and $y = y_2$, this means that the derivative of $y_2(x)$ does not exceed that of $y_1(x)$. In other words, the difference $y_2(x) - y_1(x)$ is a non-increasing function of x. But this difference is bounded from below, since $y_1(x) \leqslant y_2(x)$. This proves the existence of the limit claimed in (iii).

Clearly, (III) follows from (iii).

DISCONTINUOUS LIMITS AND FOURIER-STIELTJES INTEGRALS

By A. P. GUINAND (*Oxford*)

[Received 20 June 1946]

1. Introduction

In a recent paper† I pointed out that, if the Riemann hypothesis is true, and we write

$$a_n = -1, \qquad \alpha_n = \gamma_n$$

$$b_n = \frac{\log p}{(2\pi)^{\frac{1}{2}} p^{\frac{1}{2}m}}, \qquad \beta_n = m \log p, \tag{1.1}$$

where $\frac{1}{2} \pm i\gamma_n$ runs through the non-trivial zeros of the Riemann zeta function, p runs through the prime numbers, and m through the positive integers, then for $y > 0$

$$(2\pi)^{\frac{1}{2}} \lim_{T \to \infty} \frac{1}{T} \left\{ \sum_{0 < \alpha_n < T} a_n \cos \alpha_n y \right\} = \begin{cases} b_n & (y = \beta_n), \\ 0 & (\text{elsewhere}). \end{cases} \tag{1.2}$$

I also proved that, if a further hypothesis‡ is true, then

$$(2\pi)^{\frac{1}{2}} \lim_{T \to \infty} \frac{1}{T} \left\{ \sum_{0 < \beta_n < T} b_n \cos \beta_n y - \int_0^T \cos yt \, dB(t) \right\} = \begin{cases} a_n & (y = \alpha_n), \\ 0 & (\text{elsewhere}), \end{cases} \tag{1.3}$$

where
$$B(t) = \left(\frac{2}{\pi}\right)^{\frac{1}{2}} e^{\frac{1}{2}t}.$$

Thus, if these hypotheses are true, we have a pair of reciprocal formulae (1.2) and (1.3) connecting the sequences (1.1) with each other. Another pair of such reciprocal formulae is obtained by putting

$$a_n = b_n = 1, \qquad \alpha_n = \beta_n = (2\pi)^{\frac{1}{2}} n, \qquad B(t) = 0$$

in (1.2) and (1.3). If we also put

$$T = (2\pi)^{\frac{1}{2}} N, \qquad y = (2\pi)^{\frac{1}{2}} z,$$

† *Proc. London Math. Soc.* (to appear shortly). The formula (1.2) follows immediately from a result proved by Landau. Cf. E. C. Titchmarsh, *The Zeta-function of Riemann* (Cambridge, 1930), Theorem 42.

‡ If it is true that, as $x \to \infty$

$$\sum_{p^m < x} \log p - x = o(x^{\frac{1}{2}} \log x).$$

The best result yet deduced from the Riemann hypothesis is that this expression is $O(x^{\frac{1}{2}} \log^2 x)$. Cf. A. E. Ingham, *The Distribution of the Prime Numbers* (Cambridge, 1932), 83.

then both (1.2) and (1.3) reduce to the trivial result

$$\lim_{N\to\infty}\frac{1}{N}\left\{\sum_{n=1}^{N}\cos 2\pi nz\right\}=\begin{cases}1 & (z\text{ an integer}),\\ 0 & (\text{elsewhere}).\end{cases} \tag{1.4}$$

Thus (1.4) can be regarded as a self-reciprocal example of such a reciprocity.

In the present paper I give further self-reciprocal examples of a similar type of discontinuous limit formula. For instance, I prove that

$$(2\pi)^{\frac{1}{2}}\lim_{T\to\infty}\frac{1}{T}\left\{\sum_{0<\alpha_n<T}a_n\cos(\alpha_n y-\phi)\right\}=\begin{cases}a_n & (y=\alpha_n),\\ 0 & (\text{elsewhere}),\end{cases} \tag{1.5}$$

where $\quad a_n=n^{-\frac{1}{4}}r(n),\qquad \alpha_n=(2\pi n)^{\frac{1}{2}},\qquad \phi=\frac{1}{4}\pi, \quad$ (1.6)

and $r(n)$ is the number of ways of expressing n as the sum of the two squares.

The chief interest of these examples is that, unlike (1.1), (1.2), and (1.3), they involve no unproved hypotheses, and, unlike (1.4), they are not trivial.

In the final section of the paper I discuss the connexion between such discontinuous limits and certain types of Fourier-Stieltjes integrals.

2. First example

I first prove the example stated in (1.5) and (1.6). I use the following summation formula:†

If $f(x)$ is continuous and of bounded variation in $(0,T)$, then

$$\sum_{0\leqslant n\leqslant T}{}' r(n)f(n)=\sum_{n=0}^{\infty}r(n)g(n), \tag{2.1}$$

where $\quad g(x)=\pi\int_0^T f(t)J_0(2\pi x^{\frac{1}{2}}t^{\frac{1}{2}})\,dt, \quad$ (2.2)

and the dash indicates that the term $n=T$ is to be halved if T is an integer.

If $z>0$ and we put

$$f(x)=\pi J_0(2\pi x^{\frac{1}{2}}z^{\frac{1}{2}})$$

† E. Landau, *Vorlesungen über Zahlentheorie*, ii (Leipzig, 1927), 274.

in (2.2), then,† for $x \neq z$,

$$g(x) = \pi^2 \int_0^T J_0(2\pi z^{\frac{1}{2}}t^{\frac{1}{2}})J_0(2\pi x^{\frac{1}{2}}t^{\frac{1}{2}})\,dt$$

$$= \frac{\pi T^{\frac{1}{2}}}{x-z}\{x^{\frac{1}{2}}J_0(2\pi z^{\frac{1}{2}}T^{\frac{1}{2}})J_1(2\pi x^{\frac{1}{2}}T^{\frac{1}{2}}) - z^{\frac{1}{2}}J_1(2\pi z^{\frac{1}{2}}T^{\frac{1}{2}})J_0(2\pi x^{\frac{1}{2}}T^{\frac{1}{2}})\},$$

and

$$g(z) = \pi^2 \int_0^T \{J_0(2\pi z^{\frac{1}{2}}t^{\frac{1}{2}})\}^2\,dt = \pi^2 T[\{J_0(2\pi z^{\frac{1}{2}}T^{\frac{1}{2}})\}^2 + \{J_1(2\pi z^{\frac{1}{2}}T^{\frac{1}{2}})\}^2].$$

Further $\quad f(0) = \pi, \qquad g(0) = \pi\left(\frac{T}{z}\right)^{\frac{1}{2}} J_1(2\pi z^{\frac{1}{2}}T^{\frac{1}{2}}).$

Hence (2.1) becomes

$$\pi + \pi \sum_{1\leqslant n\leqslant T}' r(n)J_0(2\pi n^{\frac{1}{2}}z^{\frac{1}{2}}) = \pi\left(\frac{T}{z}\right)^{\frac{1}{2}} J_1(2\pi z^{\frac{1}{2}}T^{\frac{1}{2}}) +$$

$$+\pi T^{\frac{1}{2}} \sum_{\substack{n=1\\ n\neq z}}^{\infty} \frac{r(n)}{n-z}\{n^{\frac{1}{2}}J_0(2\pi z^{\frac{1}{2}}T^{\frac{1}{2}})J_1(2\pi n^{\frac{1}{2}}T^{\frac{1}{2}}) - z^{\frac{1}{2}}J_1(2\pi z^{\frac{1}{2}}T^{\frac{1}{2}})J_0(2\pi n^{\frac{1}{2}}T^{\frac{1}{2}})\} +$$

$$+ r(z)\pi^2 T[\{J_0(2\pi z^{\frac{1}{2}}T^{\frac{1}{2}})\}^2 + \{J_1(2\pi z^{\frac{1}{2}}T^{\frac{1}{2}})\}^2], \tag{2.3}$$

where $r(z) = 0$ when z is not an integer.

Similarly, putting $f(x) = 1$ in (2.1), we find that‡

$$1 + \sum_{1\leqslant n\leqslant T}' r(n) = \pi T + T^{\frac{1}{2}} \sum_{n=1}^{\infty} \frac{r(n)}{n^{\frac{1}{2}}} J_1(2\pi n^{\frac{1}{2}}T^{\frac{1}{2}}). \tag{2.4}$$

Multiplying (2.4) by $\pi J_0(2\pi z^{\frac{1}{2}}T^{\frac{1}{2}})$ and subtracting the result from (2.3) we find that

$$\pi \sum_{0\leqslant n\leqslant T}' r(n)J_0(2\pi n^{\frac{1}{2}}z^{\frac{1}{2}}) - \pi\left(\frac{T}{z}\right)^{\frac{1}{2}} J_1(2\pi z^{\frac{1}{2}}T^{\frac{1}{2}}) -$$

$$-\pi J_0(2\pi z^{\frac{1}{2}}T^{\frac{1}{2}})\left\{\sum_{0\leqslant n\leqslant T}' r(n) - \pi T\right\} \tag{2.5}$$

$$= \pi T^{\frac{1}{2}} z J_0(2\pi z^{\frac{1}{2}}T^{\frac{1}{2}}) \sum_{\substack{n=1\\ n\neq z}}^{\infty} \frac{r(n)}{n^{\frac{1}{2}}(n-z)} J_1(2\pi n^{\frac{1}{2}}T^{\frac{1}{2}}) - \tag{2.6}$$

$$-\pi T^{\frac{1}{2}} z^{\frac{1}{2}} J_1(2\pi z^{\frac{1}{2}}T^{\frac{1}{2}}) \sum_{\substack{n=1\\ n\neq z}}^{\infty} \frac{r(n)}{n-z} J_0(2\pi n^{\frac{1}{2}}T^{\frac{1}{2}}) + \tag{2.7}$$

$$+ r(z)\pi^2 T[\{J_0(2\pi z^{\frac{1}{2}}T^{\frac{1}{2}})\}^2 + \{J_1(2\pi z^{\frac{1}{2}}T^{\frac{1}{2}})\}^2] - \tag{2.8}$$

$$- r(z)\pi\left(\frac{T}{z}\right)^{\frac{1}{2}} J_0(2\pi z^{\frac{1}{2}}T^{\frac{1}{2}})J_1(2\pi z^{\frac{1}{2}}T^{\frac{1}{2}}). \tag{2.9}$$

† See G. N. Watson, *Theory of Bessel Functions* (Cambridge, 1944), 134–5, for the required integrals.

‡ This is a well-known result. Cf. E. Landau, loc. cit. 189.

Now,† as $T \to \infty$,

$$\sum_{0 \leqslant n \leqslant T} r(n) - \pi T = O(T^{\frac{1}{3}}), \tag{2.10}$$

and,‡ as $w \to \infty$,

$$J_\mu(w) = \left(\frac{2}{\pi w}\right)^{\frac{1}{2}} \left\{\cos(w - \tfrac{1}{2}\mu\pi - \tfrac{1}{4}\pi) - \frac{4\mu^2 - 1}{8w} \sin(w - \tfrac{1}{2}\mu\pi - \tfrac{1}{4}\pi)\right\} + $$
$$+ O(w^{-\frac{5}{2}}) \tag{2.11}$$
$$= O(w^{-\frac{1}{2}}).$$

Hence, as $T \to \infty$, the term (2.5) is of order $O(T^{\frac{1}{12}})$. Further (2.6) is of order

$$O\left\{T^{\frac{1}{4}} \sum_{\substack{n=1 \\ n \neq z}}^{\infty} \frac{r(n) T^{-\frac{1}{4}}}{n^{\frac{3}{4}} |n - z|}\right\} = O\left\{\sum_{n=1}^{\infty} n^{-\frac{7}{4}} r(n)\right\} = O(1),$$

and similarly (2.7) is also of order $O(1)$. Now, by (2.11), as $w \to \infty$,

$$\{J_0(w)\}^2 + \{J_1(w)\}^2 = \frac{2}{\pi w}\{\cos^2(w - \tfrac{1}{4}\pi) + \cos^2(w - \tfrac{3}{4}\pi)\} + O(w^{-2})$$
$$= \frac{2}{\pi w} + O(w^{-2}),$$

and hence (2.8) is equal to

$$\left(\frac{T}{z}\right)^{\frac{1}{2}} r(z) + O(1).$$

Finally (2.9) is of order $O(1)$, and we have, altogether,§

$$\pi \sum_{0 \leqslant n \leqslant T} r(n) J_0(2\pi n^{\frac{1}{2}} z^{\frac{1}{2}}) - \pi \left(\frac{T}{z}\right)^{\frac{1}{2}} J_1(2\pi z^{\frac{1}{2}} T^{\frac{1}{2}}) = \left(\frac{T}{z}\right)^{\frac{1}{2}} r(z) + O(T^{\frac{1}{12}}). \tag{2.12}$$

Hence
$$\lim_{T \to \infty} \pi \left(\frac{z}{T}\right)^{\frac{1}{2}} \left\{\sum_{0 \leqslant n \leqslant T} r(n) J_0(2\pi n^{\frac{1}{2}} z^{\frac{1}{2}})\right\} = r(z). \tag{2.13}$$

† E. Landau, loc. cit. 204–6. A more precise result is proved by L. K. Hua, *Quart. J. of Math.* (Oxford), 13 (1942), 18–29.

‡ G. N. Watson, loc. cit. 195.

§ It has also been proved that the series

$$\sum_{n=0}^{\infty} r(n) J_0(2\pi n^{\frac{1}{2}} z^{\frac{1}{2}})$$

is summable (C, k) to zero if $k > \frac{1}{2}$ and $r(z) = 0$. Cf. A. L. Dixon and W. L. Ferrar, *Quart. J. of Math.* (Oxford), 5 (1934), 172–85, Theorem 1.

Now, by (2.11),

$$J_0(w)-\frac{1}{8w}J_1(w)$$

$$=\left(\frac{2}{\pi w}\right)^{\frac{1}{2}}\left\{\cos(w-\tfrac{1}{4}\pi)+\frac{1}{8w}\sin(w-\tfrac{1}{4}\pi)-\frac{1}{8w}\cos(w-\tfrac{3}{4}\pi)\right\}+O(w^{-\frac{5}{2}})$$

$$=\left(\frac{2}{\pi w}\right)^{\frac{1}{2}}\cos(w-\tfrac{1}{4}\pi)+O(w^{-\frac{5}{2}}).$$

Hence

$$\sum_{1\leqslant n\leqslant T} r(n)J_0(2\pi n^{\frac{1}{2}}z^{\frac{1}{2}}) = \frac{1}{\pi z^{\frac{1}{4}}}\sum_{1\leqslant n\leqslant T}\frac{r(n)}{n^{\frac{1}{4}}}\cos(2\pi n^{\frac{1}{2}}z^{\frac{1}{2}}-\tfrac{1}{4}\pi)+$$

$$+\frac{1}{8\pi^2 z^{\frac{1}{2}}}\sum_{1\leqslant n\leqslant T}\frac{r(n)}{n^{\frac{1}{2}}}J_1(2\pi n^{\frac{1}{2}}z^{\frac{1}{2}})+O\left\{z^{-\frac{5}{4}}\sum_{1\leqslant n\leqslant T}\frac{r(n)}{n^{\frac{5}{4}}}\right\}$$

$$=\frac{1}{\pi z^{\frac{1}{4}}}\sum_{1\leqslant n\leqslant T}\frac{r(n)}{n^{\frac{1}{4}}}\cos(2\pi n^{\frac{1}{2}}z^{\frac{1}{2}}-\tfrac{1}{4}\pi)+O(1),$$

since the second series in the right-hand side converges, as $T\to\infty$, by (2.4). Hence (2.13) becomes

$$\lim_{T\to\infty}\frac{z^{\frac{1}{4}}}{T^{\frac{1}{2}}}\left\{\sum_{1\leqslant n\leqslant T}\frac{r(n)}{n^{\frac{1}{4}}}\cos(2\pi n^{\frac{1}{2}}z^{\frac{1}{2}}-\tfrac{1}{4}\pi)\right\}=r(z). \tag{2.14}$$

If we put $a_n = n^{-\frac{1}{4}}r(n)$, $\alpha_n=(2\pi n)^{\frac{1}{2}}$, $y=(2\pi z)^{\frac{1}{2}}$, $U=(2\pi T)^{\frac{1}{2}}$, then (2.14) becomes

$$(2\pi)^{\frac{1}{2}}\lim_{U\to\infty}\frac{1}{U}\left\{\sum_{0<\alpha_n<U}a_n\cos(\alpha_n y-\tfrac{1}{4}\pi)\right\}=\begin{cases}a_n & (y=\alpha_n),\\ 0 & (\text{elsewhere}),\end{cases}$$

as required.

3. Further examples

More examples of such discontinuous limits can be derived from other summation formulae. Proofs are omitted since they do not differ in principle from that in the preceding section. In each case I merely quote the results corresponding to (2.12) and (1.5).

(A) If $d(n)$ is the number of divisors† of n, and C is Euler's constant, then

$$-2\pi\sum_{1\leqslant n\leqslant T}d(n)Y_0(4\pi n^{\frac{1}{2}}z^{\frac{1}{2}})+\left(\frac{T}{z}\right)^{\frac{1}{2}}(\log T+2C)Y_1(4\pi z^{\frac{1}{2}}T^{\frac{1}{2}})$$

$$=O(T^{\frac{1}{12}})+2\left(\frac{T}{z}\right)^{\frac{1}{2}}\begin{cases}d(z) & (z\text{ an integer}),\\ 0 & (\text{elsewhere}).\end{cases}$$

† The proof requires a special form of Voronoï's summation-formula for functions with logarithmic singularities at the origin. Cf. A. L. Dixon and W. L. Ferrar, *Quart. J. of Math.* (Oxford), 8 (1937), 66–74, Theorem 3.

Further, (1.5) holds with $a_n = n^{-\frac{1}{4}}d(n)$, $\alpha_n = (4\pi n)^{\frac{1}{2}}$, $\phi = -\frac{1}{4}\pi$.

(B) Similarly†

$$\tfrac{1}{2}\pi \sum_{1\leqslant 2n+1\leqslant T} (-1)^n d(2n+1) Y_0\{\pi(2n+1)^{\frac{1}{2}}z^{\frac{1}{2}}\}$$

$$= O(T^{\frac{1}{12}})+\frac{1}{2}\left(\frac{T}{z}\right)^{\frac{1}{2}}\begin{cases}(-1)^{\frac{1}{2}(z-1)}\, d(z) & (z \text{ an odd integer}),\\ 0 & (\text{elsewhere}).\end{cases}$$

Further, (1.5) holds with

$$a_n = (-1)^n(2n+1)^{-\frac{1}{4}}d(2n+1), \qquad \alpha_n = (2n+1)^{\frac{1}{2}}\pi^{\frac{1}{2}}, \qquad \phi = \tfrac{3}{4}\pi.$$

(C) If $r_p(n)$ is the number of ways of expressing n as the sum of p squares,‡ then for $p = 3$ or 4 and for any positive ϵ

$$\pi \sum_{1\leqslant n\leqslant T}^{\infty} \frac{r_p(n)}{n^{\frac{1}{4}p-\frac{1}{2}}} J_{\frac{1}{2}p-1}(2\pi n^{\frac{1}{2}}z^{\frac{1}{2}}) - \frac{\pi^{\frac{1}{2}p}T^{\frac{1}{2}p}}{z^{\frac{1}{2}}\Gamma(\frac{1}{2}p)} J_{\frac{1}{2}p}(2\pi z^{\frac{1}{2}}T^{\frac{1}{2}})$$

$$= O(T^{\kappa+\epsilon})+\left(\frac{T}{z}\right)^{\frac{1}{2}}\begin{cases}\dfrac{r_p(z)}{z^{\frac{1}{4}p-\frac{1}{2}}} & (z \text{ an integer}),\\ 0 & (\text{elsewhere}),\end{cases}$$

where $\kappa = \frac{7}{29}$ when $p = 3$ and $\kappa = \frac{1}{4}$ when $p = 4$. Corresponding to (1.5) we find that

$$(2\pi)^{\frac{1}{2}} \lim_{T\to\infty} \frac{1}{T}\Bigg\{\sum_{0<\alpha_n<T} a_n \cos(\alpha_n y - \tfrac{1}{4}p\pi+\tfrac{1}{4}\pi) -$$

$$- \frac{\pi^{\frac{1}{2}p-\frac{1}{2}}T^{\frac{1}{2}p-\frac{1}{2}}}{2^{\frac{1}{2}p-\frac{1}{2}}y\Gamma(\frac{1}{2}p)}\sin(yT-\tfrac{1}{4}p\pi+\tfrac{1}{4}\pi)\Bigg\}$$

$$= \begin{cases} a_n & (y = \alpha_n),\\ 0 & (\text{elsewhere}),\end{cases}$$

where $a_n = n^{-\frac{1}{4}p+\frac{1}{4}}r_p(n)$, $\alpha_n = (2\pi n)^{\frac{1}{2}}$.

(D) If $\tau(n)$ is Ramanujan's arithmetical function,§ defined by

$$x\{(1-x)(1-x^2)(1-x^3)...\}^{24} = \sum_{n=1}^{\infty} \tau(n)x^n, \quad |x| < 1,$$

† The required summation-formula is given by A. P. Guinand, *Quart. J. of Math.* (Oxford), 9 (1938), 53–67, Theorem 6.

‡ Cf. idem, ibid. 10 (1939), 104–18, Theorem 5, for the required summation-formula, and A. Walfisz, *Math. Annalen*, 95 (1926), 69–83, J. R. Wilton, *Proc. London Math. Soc.* (2), 29 (1929), 168–88, for the approximations corresponding to (2.10). In these cases the series corresponding to (2.6) and (2.7) do not all converge, and we have to use lemmas given in Wilton's paper.

§ Cf. A. P. Guinand, *Proc. Cambridge Phil. Soc.* (to appear shortly), for the required summation-formula, and R. A. Rankin, ibid. 36 (1940), 150–1, for the approximation corresponding to (2.10).

then

$$2\pi \sum_{1 \leqslant n \leqslant T} \frac{\tau(n)}{n^{\frac{11}{2}}} J_{11}(4\pi n^{\frac{1}{2}} z^{\frac{1}{2}}) = O(T^{\frac{3}{20}}) + 2\left(\frac{T}{z}\right)^{\frac{1}{2}} \begin{cases} \dfrac{\tau(z)}{z^{\frac{11}{2}}} & (z \text{ an integer}), \\ 0 & (\text{elsewhere}). \end{cases}$$

Further, (1.5) holds with $a_n = n^{-\frac{23}{4}}\tau(n)$, $\alpha_n = (4\pi n)^{\frac{1}{2}}$, $\phi = -\frac{1}{4}\pi$.

(E) For the sake of completeness I also note the following extensions of the trivial formula (1.4) to sums involving primitive characters.†

If $\chi(n)$ is a real primitive character modulo k ($k \geqslant 1$) then

$$\lim_{T\to\infty} \frac{k^{\frac{1}{2}}}{T} \left\{ \sum_{1 \leqslant n \leqslant T} \chi(n) \begin{matrix}\cos\\ \sin\end{matrix} \left(\frac{2\pi n z}{k}\right) \right\} = \begin{cases} \chi(z) & (z \text{ an integer}), \\ 0 & (\text{elsewhere}), \end{cases}$$

where the cosine or the sine is taken according as $\chi(-1)$ is equal to $+1$ or -1.

4. Fourier-Stieltjes integrals

In this section I prove some general theorems showing how the discontinuous limits of the previous sections are related to certain Fourier-Stieltjes integrals.

It should first be noted that we cannot expect to prove that a result of the form (1.2) directly implies a result of the form (1.3). In any given example of such a reciprocity we can omit any finite number of terms a_n from (1.2) without affecting the sequence $\{b_n\}$; thus the omitted terms a_n still arise in (1.3), and the reciprocity no longer holds.

Also, it is clear from the argument of § 2 that the phase-angle ϕ in the various examples of § 3 depends on the type and order of Bessel functions in the associated summation-formula. For simplicity I only discuss in detail the case corresponding to summation-formulae involving the Fourier cosine-transformation, when $\phi = 0$.

I first show how a Fourier-Stieltjes inversion formula‡ can be deduced from the Hankel inversion formula of order $\frac{3}{2}$.

It is sufficient for our purposes to define the Stieltjes integral by

$$\int_a^b u(x)\, dv(x) = \big[u(x)v(x)\big]_a^b - \int_a^b v(x)u'(x)\, dx,$$

where the latter integral is a Lebesgue integral.

† Cf. E. Landau, *Handbuch der Lehre von der Verteilung der Primzahlen*, 1 (Leipzig, 1909), chapters xxii, xxx for the required properties of primitive characters.

‡ Cf. J. C. Burkill, *Proc. London Math. Soc.* (2), 25 (1925), 513–24, for the convergence theory of such inversions.

THEOREM 1. *If $f(x)/x$ belongs to $L^2(0,\infty)$, $f(x)$ tends to zero as $x \to +0$ and is $o(x^2)$ at infinity, then there exists a function $g(x)$ such that $g(x)/x$ belongs to $L^2(0,\infty)$, and*

$$\int_0^y g(x)\,dx = \left(\frac{2}{\pi}\right)^{\frac{1}{2}} \int_0^\infty \frac{1-\cos xy}{x^2}\,df(x); \tag{4.1}$$

if $g(x)$ also satisfies the same conditions as $f(x)$ at the origin and infinity, then

$$\int_0^y f(x)\,dx = \left(\frac{2}{\pi}\right)^{\frac{1}{2}} \int_0^\infty \frac{1-\cos xy}{x^2}\,dg(x). \tag{4.2}$$

Further, if $f(x)$ is of bounded variation in a neighbourhood of $x = y$ and $g(x)$ is $o(x)$ at infinity, then

$$\tfrac{1}{2}\{f(y+0)+f(y-0)\} = \left(\frac{2}{\pi}\right)^{\frac{1}{2}} \int_0^{\to\infty} \frac{\sin xy}{x}\,dg(x). \tag{4.3}$$

Let $g(x)/x$ be the Hankel transform† of order $\frac{3}{2}$ of $f(x)/x$. Then $g(x)/x$ belongs to $L^2(0,\infty)$. Further, the Hankel transform of order $\frac{3}{2}$ of the function

$$F(x) = \begin{cases} x & (x < y), \\ 0 & (x > y), \end{cases}$$

is

$$\begin{aligned}\int^y (xt)^{\frac{1}{2}} J_{\frac{3}{2}}(xt)t\,dt &= \left(\frac{2}{\pi}\right)^{\frac{1}{2}} \int_0^y \left(\frac{\sin xt}{x} - t\cos xt\right) dt \\ &= \left(\frac{2}{\pi}\right)^{\frac{1}{2}} \left\{\frac{2}{x^2}(1-\cos xy) - \frac{y}{x}\sin xy\right\} \\ &= -x\left(\frac{2}{\pi}\right)^{\frac{1}{2}} \frac{d}{dx}\left(\frac{1-\cos xy}{x^2}\right).\end{aligned}$$

Hence, by the Parseval theorem for these transforms,

$$\begin{aligned}\int_0^y g(x)\,dx &= -\left(\frac{2}{\pi}\right)^{\frac{1}{2}} \int_0^\infty \frac{f(x)}{x} x \frac{d}{dx}\left(\frac{1-\cos xy}{x^2}\right) dx \\ &= -\left(\frac{2}{\pi}\right)^{\frac{1}{2}} \left[f(x)\frac{1-\cos xy}{x^2}\right]_0^\infty + \left(\frac{2}{\pi}\right)^{\frac{1}{2}} \int_0^\infty \frac{1-\cos xy}{x^2}\,df(x).\end{aligned}$$

The integrated terms vanish by the assumptions on $f(x)$, and we have (4.1). The inverse formula (4.2) follows in the same way.

† E. C. Titchmarsh, *Fourier Integrals* (Oxford, 1937), 214 and 240.

Now, if $f(x)$ is of bounded variation in a neighbourhood of $x = y$, then†

$$\frac{1}{2y}\{f(y+0)+f(y-0)\} = \left(\frac{2}{\pi}\right)^{\frac{1}{2}} \int_0^{\to\infty} (xy)^{\frac{1}{2}} J_{\frac{3}{2}}(xy) \frac{g(x)}{x}\, dx$$

$$= -\left(\frac{2}{\pi}\right)^{\frac{1}{2}} \int_0^{\to\infty} g(x) \frac{d}{dx}\left(\frac{\sin xy}{xy}\right) dx$$

$$= -\left(\frac{2}{\pi}\right)^{\frac{1}{2}} \left[g(x)\frac{\sin xy}{xy}\right]_0^{\to\infty} + \left(\frac{2}{\pi}\right)^{\frac{1}{2}} \int_0^{\to\infty} \frac{\sin xy}{xy}\, dg(x),$$

and (4.3) follows immediately.

THEOREM 2. *If* (i) $f(x)$ *and* $g(x)$ *are a pair of transforms in the sense of Theorem* 1, *and are both* $O(x^{\frac{1}{2}})$ *at infinity, and* (ii) $F(x)$ *is an integral, tends to zero at infinity, and* $xF'(x)$ *belongs to* $L^2(0,\infty)$, *then*

$$G(x) = \left(\frac{2}{\pi}\right)^{\frac{1}{2}} \int_0^{\to\infty} F(y) \cos xy\, dy \tag{4.4}$$

converges for $x > 0$ *and*

$$\int_0^{\infty} F(x)\, df(x) = \int_0^{\infty} G(x)\, dg(x). \tag{4.5}$$

It has been proved that, with these conditions on $F(x)$, the function $G(x)$ is also an integral and $xF'(x)$, $xG'(x)$ are a pair of Hankel transforms of order $\frac{3}{2}$, and that $F(x)$ and $G(x)$ are $o(x^{-\frac{1}{2}})$ as x tends to zero or to infinity.‡ Hence, by the Parseval theorem for Hankel transforms of order $\frac{3}{2}$,

$$\int_0^{\infty} \frac{f(x)}{x} xF'(x)\, dx = \int_0^{\infty} \frac{g(x)}{x} xG'(x)\, dx.$$

When we integrate by parts, the integrated terms vanish, and (4.5) follows immediately.

THEOREM 3. *If* $f(x)$ *and* $g(x)$ *satisfy the conditions of Theorems* 1 *and* 2, *and the only discontinuities of* $g(x)$ *are simple discontinuities at*

† E. C. Titchmarsh, *Fourier Integrals* (Oxford, 1937), 83 and 266.

‡ A. P. Guinand, *Annals of Math.* 42 (1941), 591–603, Lemmas 2 and 4 with $p = 2$.

a discrete set of points, and elsewhere $g(x)$ has a derivative $g'(x)$ which is of bounded variation in any finite interval not including a point of discontinuity of $g(x)$, then

$$(2\pi)^{\frac{1}{2}} \lim_{T\to\infty} \frac{1}{T} \int_0^T \cos xy \, df(x) = g(y+0)-g(y-0). \tag{4.6}$$

Put

$$F(x) = \begin{cases} \frac{(2\pi)^{\frac{1}{2}}}{T}(\cos xy - \cos yT) & (x \leqslant T), \\ 0 & (x \geqslant T), \end{cases}$$

in Theorem 2. Then, for $x \neq y$,

$$\begin{aligned} G(x) &= \frac{\sin(x+y)T}{(x+y)T} + \frac{\sin(x-y)T}{(x-y)T} - \frac{2}{xT}\sin xT \cos yT \qquad (4.7) \\ &= \frac{2y}{Tx(y^2-x^2)}(x\cos xT \sin yT - y \sin xT \cos yT), \end{aligned}$$

and

$$G(y) = 1 - \frac{\sin 2yT}{2yT}. \tag{4.8}$$

Hence, for any fixed positive δ,

$$\begin{aligned} &\int_0^{y-\delta} G(x)\, dg(x) \\ &= \left[\frac{2yg(x)}{Tx(y^2-x^2)}(x\cos xT \sin yT - y\sin xT\cos yT)\right]_0^{y-\delta} - \\ &-2y \int_0^{y-\delta} \frac{g(x)}{x(y^2-x^2)}\left\{\frac{\cos xT \sin yT}{T} - x\sin xT \sin yT - y\cos xT \cos yT + \right. \\ &\qquad \left. + \frac{3x^2-y^2}{Tx(y^2-x^2)}(x\cos xT\sin yT - y\sin xT\cos yT)\right\} dx \\ &= O(T^{-1}) + 2y \int_0^{y-\delta} \frac{g(x)}{x(y^2-x^2)}(x\sin xT \sin yT + y\cos xT\cos yT)\, dx. \end{aligned}$$

Now, if $k = 0$ or 1,

$$\int_0^{y-\delta} \left|\frac{g(x)}{x^k(y^2-x^2)}\right| dx \leqslant \left\{\int_0^{y-\delta} \left|\frac{g(x)}{x}\right|^2 dx\right\}^{\frac{1}{2}} \left\{\int_0^{y-\delta} \frac{x^{2-2k}\, dx}{(y^2-x^2)^2}\right\}^{\frac{1}{2}}.$$

Hence this integral converges, and, by the Riemann-Lebesgue theorem,† the integrals

$$\int_0^{y-\delta} \frac{g(x)}{y^2-x^2}\sin xT\,dx, \qquad \int_0^{y-\delta} \frac{g(x)}{x(y^2-x^2)}\cos xT\,dx$$

tend to zero as $T \to \infty$. Hence, as $T \to \infty$,

$$\int_0^{y-\delta} G(x)\,dg(x) = o(1). \tag{4.9}$$

Similarly
$$\int_{y+\delta}^{\infty} G(x)\,dg(x) = o(1). \tag{4.10}$$

Now consider
$$\int_{y-\delta}^{y+\delta} G(x)\,dg(x).$$

Suppose δ chosen so small that the interval $y-\delta \leqslant x \leqslant y+\delta$ contains no discontinuity of $g(x)$, except possibly at $x = y$. Then, by (4.8),

$$\int_{y-\delta}^{y+\delta} G(x)\,dg(x) = \{g(y+0)-g(y-0)\}\left\{1-\frac{\sin 2yT}{2yT}\right\}+\int_{y-\delta}^{y+\delta} G(x)g'(x)\,dx. \tag{4.11}$$

Now, by (4.7), in the interval $(y-\delta,\ y+\delta)$

$$G(x) = O(T^{-1})+\frac{\sin(x-y)T}{(x-y)T}.$$

Hence (4.11) becomes

$$\int_{y-\delta}^{y+\delta} G(x)\,dg(x) = g(y+0)-g(y-0)+O(T^{-1})+$$
$$+\frac{1}{T}\int_{y-\delta}^{y+\delta} g'(x)\frac{\sin(x-y)T}{x-y}\,dx. \tag{4.12}$$

By Fourier's single-integral theorem‡

$$\lim_{T\to\infty}\int_{y-\delta}^{y+\delta} g'(x)\frac{\sin(x-y)T}{x-y}\,dx = \tfrac{1}{2}\pi\{g'(y+0)+g'(y-0)\}.$$

Hence the last term in (4.12) is also $O(T^{-1})$, and, combining this result with (4.9) and (4.10) we see that

$$\int_0^{\infty} G(x)\,dg(x) = g(y+0)-g(y-0)+o(1) \tag{4.13}$$

as $T \to \infty$.

† E. C. Titchmarsh, *Fourier Integrals* (Oxford, 1937), 11. ‡ Ibid. 25.

Further

$$\int_0^\infty F(x)\,df(x) = \frac{(2\pi)^{\frac{1}{2}}}{T}\int_0^T (\cos xy - \cos yT)\,df(x)$$

$$= \frac{(2\pi)^{\frac{1}{2}}}{T}\int_0^T \cos xy\,df(x) - \frac{(2\pi)^{\frac{1}{2}}}{T} f(T)\cos yT$$

$$= \frac{(2\pi)^{\frac{1}{2}}}{T}\int_0^T \cos xy\,df(x) + O(T^{-\frac{1}{2}}) \tag{4.14}$$

since $f(T) = O(T^{\frac{1}{2}})$ by assumption (i) of Theorem 2. The required result (4.6) follows from (4.13), (4.14), and Theorem 2.

THEOREM 4. *If $f(x)$ and $g(x)$ are a pair of transforms both satisfying the conditions of Theorems 1, 2, 3, and*

$$f(x) = \tfrac{1}{2}\{f(x+0)+f(x-0)\},$$

$$g(x) = \tfrac{1}{2}\{g(x+0)+g(x-0)\},$$

then $f(x)$ and $g(x)$ can be expressed in the forms

$$f(x) = \sum_{0\leqslant\alpha_n\leqslant x}{}' a_n - A(x), \tag{4.15}$$

$$g(x) = \sum_{0\leqslant\beta_n\leqslant x}{}' b_n - B(x), \tag{4.16}$$

where $A(x)$ and $B(x)$ are continuous functions of x. Further

$$(2\pi)^{\frac{1}{2}}\lim_{T\to\infty}\frac{1}{T}\left\{\sum_{0<\alpha_n<T} a_n\cos\alpha_n y - \int_0^T \cos xy\,dA(x)\right\} = \begin{cases} b_n & (y=\beta_n),\\ 0 & (elsewhere),\end{cases} \tag{4.17}$$

and

$$(2\pi)^{\frac{1}{2}}\lim_{T\to\infty}\frac{1}{T}\left\{\sum_{0<\beta_n<T} b_n\cos\beta_n y - \int_0^T \cos xy\,dB(x)\right\} = \begin{cases} a_n & (y=\alpha_n),\\ 0 & (elsewhere).\end{cases} \tag{4.18}$$

If α_n runs through the discrete set of points of discontinuity of $f(x)$ and we put

$$a_n = f(\alpha_n+0) - f(\alpha_n-0),$$

then the function

$$A(x) = \sum_{0\leqslant\alpha_n\leqslant x}{}' a_n - f(x)$$

has no discontinuities, as required. By substituting (4.15) in (4.6), the formula (4.17) follows immediately, and (4.16) and (4.18) follow in the same way.

Thus we see that discontinuous limits of the type discussed in the previous sections are associated with pairs of Fourier-Stieltjes transforms which both have discrete sets of points of simple discontinuity. For example, the function

$$f(x) = \left[\frac{x}{(2\pi)^{\frac{1}{2}}}\right] - \frac{x}{(2\pi)^{\frac{1}{2}}}$$

is self-reciprocal with respect to the transformation (4.3). Substituting this function in Theorem 4 we find that (4.17) and (4.18) reduce to (1.4).

Similar results can also be proved for Hankel-Stieltjes integrals. With appropriate modifications of the conditions on $f(x)$ and $g(x)$ we find that for $\nu > 0$

$$\pi \lim_{T\to\infty} \frac{y^\nu}{T} \int\limits_0^T x^{1-\nu} J_{\nu-1}(xy)\, df(x) = g(y+0) - g(y-0), \tag{4.19}$$

where
$$g(x) = \int\limits_0^{\to\infty} \left(\frac{x}{t}\right)^\nu J_\nu(xt)\, df(t). \tag{4.20}$$

For example, if $\nu = 1$, the function

$$f(x) = \sum_{1\leqslant n\leqslant x^2/2\pi} r(n) - \tfrac{1}{2}x^2$$

is self-reciprocal with respect to the transformation (4.20), and (4.19) reduces to (2.13).

SETS OF NON-INTEGRAL FUNCTIONAL POWERS

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1. Introduction

In a recent paper* a formal definition of non-integral functional powers was given, and a *power-set* of functions was defined. In the present paper I prove some general theorems on power-sets, and describe methods of constructing certain types of functional powers and power-sets.

2. Notation and definitions

The notation used is that of F.P. We consider only functions which are continuous and strictly increasing in the closed interval $\langle a, b\rangle$, and say that f is *c.n.f.* (complete and node-free†) in $\langle a, b\rangle$ if a and b but no other points in $\langle a, b\rangle$ are nodes of f.

Definition. *A 'power-set' is a set P of functions, defined in $\langle a, b\rangle$, having the properties* (i) *every member of P is complete in $\langle a, b\rangle$*, (ii) *every member of P commutes with every other member,* (iii) *every integral power and every functional product of integral powers is either node-free in (a, b) or is the identity function $I(x) \equiv x$.*

It is clear from the definition that the set P can be assumed without loss of generality to contain all integral functional powers of each member and the functional product of every pair of members, as shown in F.P. § 5.

We choose any member f of P as base-function, where for convenience $f > I$. It can then be shown (F.P. § 5) that with any other member, say g, a real number λ can be associated. This number λ has the properties that‡ $g^n \gtreqless f^p$ according as $n\lambda \gtreqless p$. We call λ the *index* of g with respect to f, and write $g = f^\lambda$.

3. Some properties of power-sets

The following index properties, proved in F.P., are restated here for convenience.

* J. S. Batty and A. G. Walker, 'Non-integral Functional Powers', *Quart. J. of Math.* (Oxford), 17 (1946), 146–52. This paper will be referred to as F.P.

† A *node* of f is a number satisfying $f(x) = x$, and f is *complete* in $\langle a, b\rangle$ if it is continuous and strictly increasing and if a and b are nodes of f.

‡ Throughout this paper n will denote a positive integer, and p, q will denote integers which may be positive, negative, or zero.

If $g = f^{\lambda}$ and $h = f^{\mu}$, then

[3.1] $g^p h^q = f^{p\lambda + q\mu}$,

[3.2] *$g \gtreqless h$ in (a, b) according as $\lambda \gtreqless \mu$.*

In particular, $g > I$ if $\lambda > 0$. We note that g is an increasing function of λ for fixed x.

[3.3] *If $\lambda \neq 0$, we can refer to g as base-function, and then $f = g^{1/\lambda}$, $h = g^{\mu/\lambda}$.*

We denote the set of indices associated with a power-set P by Λ, members being λ, μ, ν,...; and write $P(f, \Lambda)$ for the set P, in which the base-function is f and the corresponding set of indices is Λ. It is clear that Λ is a corpus (or modulus), the sum and difference of any two members being members.

The classification of power-sets was discussed in F.P. It was shown that two main types may be distinguished, according as the set Λ is discrete or not. If the set Λ is discrete, the base-function may be so chosen that all members are integral powers of this function, and the set P is called an *integral* set. Such sets need no further attention.

If Λ is not discrete, then it is everywhere dense. In this case, many interesting problems arise; for the remainder of this paper we shall be concerned only with this case.

4. The limit-function of a power-set

Definition. *The 'limit-function' $L(x)$ of a power-set $P(f, \Lambda)$ is defined by*

$$L(x) = \lim_{\lambda \to +0} f^{\lambda}(x), \tag{1}$$

where $\lambda \to +0$ in any way through values from the set Λ.

Since we have $f^{\lambda} > I$ for $\lambda > 0$, and f^{λ} decreases with λ for each x, it follows that $L(x)$ exists uniquely in $\langle a, b \rangle$; and that, for $\lambda > 0$,

$$x \leqslant L(x) \leqslant f^{\lambda}(x) \quad (a < x < b). \tag{2}$$

Also, since $f^{\lambda}(x)$ increases with x,

$$L(x) \leqslant L(x') \quad (a < x < x' < b). \tag{3}$$

By taking the limit as $\lambda \to +0$ in the relation $f^{\lambda} f^{\mu} = f^{\mu} f^{\lambda}$, we have, since $f^{\mu}(x)$ is a continuous function of x,

$$L f^{\mu} = f^{\mu} L, \tag{4}$$

i.e. L commutes with every member of the power-set.

5. Uniform power-sets

The function $f^{\lambda}(x)$ is a continuous function of x and, by [3.2], decreases to $L(x)$, for fixed x, as $\lambda \to +0$. It follows that the convergence to $L(x)$ is uniform if and only if $L(x)$ is continuous in $\langle a, b\rangle$. It will be shown later that, if $L(x) \not\equiv x$, then $L(x)$ is discontinuous. Thus the convergence is uniform if and only if $L(x) \equiv x$.

Definition. *A 'uniform' power-set is a set for which $L(x) \equiv x$ in $\langle a, b\rangle$.*

A *related* set of functions was defined by Walker* as a set in which all members can be expressed in canonical form $\psi^{-1}\alpha\psi$ simultaneously, with the same ψ. A theorem similar to the following is given in C.F. (I), § 13.

[5.1] *The necessary and sufficient condition for a power-set to be uniform is that the set is related.*

Lemma I. *If $P(f, \Lambda)$ is a uniform power-set, and $a < x_0 < b$, then the set $\{f^{\mu}(x_0)\}$, for variable μ belonging to Λ, and fixed x_0, is everywhere dense in (a, b).*

The proof is similar to that given in C.F. (I), [9.2] and is omitted.

Lemma II. *If a function $\phi(x)$ is increasing and continuous on an everywhere-dense sub-set E of $\langle a, b\rangle$, and if the values of $\phi(x)$ are everywhere dense in some interval, then there exists a unique function, increasing and continuous in $\langle a, b\rangle$, and equal to $\phi(x)$ at the points of E.*

The proof is immediate; for it is enough to define the function to be $\phi(x)$ at points of E, and to be the common value of $\lim\limits_{x\to x_1+0} \phi(x)$ and $\lim\limits_{x\to x_1-0} \phi(x)$ at points x_1 of CE. The function so obtained being unique, I shall refer to the process as the 'continuation' of $\phi(x)$, and denote the resulting function by the symbol $\phi(x)$.

We return to the main theorem. The sufficiency of the given condition is easily seen; for in a related set we have

$$L(x) = \lim_{\lambda\to+0} f^{\lambda}(x) = \lim_{\lambda\to+0} \psi^{-1}\alpha^{\lambda}\psi(x) = x,$$

since ψ^{-1} is continuous, and so the set P is uniform.

To prove the necessity of the condition, choose $\alpha > 1$ and x_0 in

* A. G. Walker, 'Commutative Functions' (I) and (II) (*Quart. J. of Math.* (Oxford), 17 (1946), 65–92). These papers will be referred to as C.F. (I) and C.F. (II).

(a, b). Define a function ψ at the set of points $x_\mu = f^\mu(x_0)$, where μ belongs to Λ, by

$$\psi(x_\mu) = \alpha^\mu. \tag{5}$$

Since α^μ and x_μ increase with μ, it follows that $\psi(x_\mu)$ is an increasing function of x_μ. Thus only simple discontinuities of ψ are possible, and these are excluded since the values α^μ of ψ are everywhere dense in $(0, \infty)$. Hence $\psi(x_\mu)$ is continuous on the set $\{x_\mu\}$. (We note also that $\psi^{-1}(\alpha^\mu)$ is continuous on the set $\{\alpha^\mu\}$). By Lemma II, $\psi(x)$ can now be continued to be continuous and increasing in (a, b). The limits $\mu \to \pm\infty$ give $\psi(a) = 0$, $\psi(b) = \infty$, as required.

From (5) we now have, for any λ of Λ,

$$\psi f^\lambda(x_\mu) = \psi f^{\lambda+\mu}(x_0) = \psi(x_{\lambda+\mu}) = \alpha^{\lambda+\mu} = \alpha^\lambda \psi(x_\mu).$$

Thus $\psi f^\lambda = \alpha^\lambda \psi$ on the everywhere-dense set $\{x_\mu\}$, and therefore throughout $\langle a, b\rangle$, by the continuity of f^λ and ψ. Hence for each λ of Λ, $f^\lambda = \psi^{-1}\alpha^\lambda\psi$, where ψ is continuous and increasing in $\langle a, b\rangle$, and $\psi(a) = 0$, $\psi(b) = \infty$. Thus the set P is related.

It is clear that a uniform set may be augmented, without inconsistency, so that the set Λ is the continuum, the function $\psi^{-1}\alpha^\lambda\psi$ being defined for all λ.

[5.2] *If $L(x) = x$ in any interval of $\langle a, b\rangle$, then $L(x) \equiv x$ in $\langle a, b\rangle$.*

The proof is similar to that given in C.F. (I), [9.1] and is omitted.

6. The limit function of a non-uniform set

I proceed to discuss power-sets for which $L(x) \not\equiv x$. Their existence follows from C.F. (II), § 16, and will be demonstrated again in § 10 below. We describe such sets as 'non-uniform', since, as will be shown in the following theorem, the convergence in this case is non-uniform, $L(x)$ being discontinuous. It will be seen that the properties of $L(x)$ are similar to those of the limit function associated with a pair of semi-related functions, described in C.F. (II), § 14.

[6.1] *The limit function $L(x)$ of a non-uniform power-set is constant in each of a set of non-overlapping closed intervals δ in $\langle a, b\rangle$. Also*

(i) *in $\delta = \langle y, z\rangle$, $L(x) = z$;*

(ii) *the set of δ-intervals is everywhere dense in $\langle a, b\rangle$, and the complementary set T is nowhere dense;*

(iii) *if t belongs to T, $L(t) = t$;*

(iv) *$L(x)$ is continuous except at the left-hand end-points of the δ-intervals.*

Take x_0 in (a, b) such that $L(x_0) \neq x_0$. Then, from (2), $L(x_0) > x_0$. Also $L(x_0) < b$, since each member of P is complete in $\langle a, b\rangle$. Choose from the everywhere-dense set Λ a positive sequence $\lambda_1, \lambda_2, \ldots$ such that

$$\lambda_n < \tfrac{1}{2}\lambda_{n-1}, \qquad \lim_{n\to\infty} \lambda_n = 0. \tag{6}$$

Write $$f^{\lambda_n} = f_n, \qquad f_n(x_0) = z_n, \qquad L(x_0) = z. \tag{7}$$

Then, by [3.2] and (6), $$f_n^2 < f_{n-1}. \tag{8}$$

Also, by (2), $z < z_n$ and so we have $x_0 < z < z_n < b$; hence, since f_n increases with x,

$$L(z) < f_n(z) < f_n(z_n) = f_n^2(x_0). \tag{9}$$

Now, by using (8),

$$f_n^2(x_0) < f_{n-1}(x_0) = z_{n-1}. \tag{10}$$

From (9) and (10), $L(z) < z_{n-1}$ for all n, and so $L(z) \leqslant z$. Also $f_n(z) > z$ gives $L(z) \geqslant z$. Hence

$$L(z) = z. \tag{11}$$

We now consider the sequence $f_n^{-1}(z)$, which, by [3.2], increases with n; write

$$y = \lim_{n\to\infty} f_n^{-1}(z). \tag{12}$$

Then clearly $y > f_n^{-1}(z)$, and we have $f_n^{-1}(z) > a$; so $y > a$. Also, since $f_n(x)$ increases with x, $y > f_n^{-1}(z)$ implies $f_n(y) > z$, whence $L(y) \geqslant z$. On the other hand, $z < z_n = f_n(x_0)$ gives $f_n^{-1}(z) < x_0$ for all n, so that $y \leqslant x_0$, whence $L(y) \leqslant L(x_0) = z$, by using (3). Combining these results we have

$$L(y) = z. \tag{13}$$

From (3), (11), (13),

$$L(x) = z \quad (y \leqslant x \leqslant z), \tag{14}$$

where $y \leqslant x_0 < z$.

I now show that $\langle y, z\rangle$ is the full extent of the interval in which $L(x) = z$. Consider any $\eta < y$. Then from the definition of y there is an n_0, such that for $n \geqslant n_0$, $\eta < f_n^{-1}(z)$, i.e. $f_n(\eta) < z$, and so $L(\eta) < z$. Also, by (3), if $\xi > z$, $L(\xi) \geqslant \xi > z$. Thus

$$L(\eta) < z, \qquad L(\xi) > z \qquad (\eta < y,\ z < \xi).$$

We have now proved that $L(x) = z$ in an interval containing x_0, the extent of the interval being $\langle y, z\rangle$, given by (7) and (12). Also $a < y < z < b$.

It follows from [5.2] that there is no interval in (a, y) or in (z, b) throughout which $L(x) \equiv x$. The above argument can therefore be

repeated at points in (a, y) and (z, b), and we deduce the existence of an everywhere-dense set of closed intervals $\langle y, z\rangle$ in (a, b), in each of which $L(x) = z$. These intervals are clearly non-overlapping since $L(x)$ is single-valued. We denote the sets of end-points, y and z, by Y and Z respectively, and the set of points x, $y < x < z$, by X.

The functions f^λ are complete in $\langle a, b\rangle$, so a cannot belong to Y nor b to Z. Also, the continuum is not the sum of a finite or infinite number of closed non-overlapping intervals. Thus there is a residual set of points in $\langle a, b\rangle$, which we denote by T. Since, if t belongs to T, we must have $L(t) = t$, it follows from [5.2] that T is nowhere dense.

Clearly $L(x)$ is discontinuous at the points of Y, but continuous elsewhere. We note that y and z are limit-points of Y, Z, and T on the left and right respectively, and t is a limit-point of each set on both sides.

7. Properties of $L(x)$ for a non-uniform set

[7.1] *The limit function $L(x)$ is not affected by a change of base function in the power-set.*

Suppose the function g to be taken as base, where $g = f^\mu$, $\mu > 0$, and denote the corresponding limit function by $L_1(x)$. Then, by [3.3], $g^\lambda = f^{\lambda\mu}$, and

$$L_1(x) = \lim_{\lambda\to+0} g^\lambda = \lim_{\lambda\to+0} f^{\lambda\mu} = \lim_{\nu\to+0} f^\nu = L(x).$$

Definition. *The 'basic set' of intervals associated with the limit function $L(x)$ of a non-uniform set is the complete set of $\langle y, z\rangle$ intervals in each of which $L(x)$ is constant.*

We note that L is completely characterized by its set of basic intervals.

[7.2] *If $a < x < b$, and λ is any member of Λ, then the points x and $f^\lambda(x)$ are members of the same point-set, X, Y, Z, or T.*

Let $\langle y_0, z_0\rangle = \delta_0$ be any member of the basic set of intervals. I prove first that $\langle f^\lambda(y_0), f^\lambda(z_0)\rangle$ is also a member. We have $L(x) = z_0$ for $y_0 \leqslant x \leqslant z_0$, and, by (4), L and f^λ commute; thus, for $y_0 \leqslant x \leqslant z_0$,

$$Lf^\lambda(x) = f^\lambda L(x) = f^\lambda(z_0),$$

i.e. $L(x)$ is constant and equal to $f^\lambda(z_0)$ throughout $\langle f^\lambda(y_0), f^\lambda(z_0)\rangle$.

It is therefore clear, since $L(x)$ is non-decreasing, that $f^\lambda(z_0)$ is a point of Z. To show that $f^\lambda(y_0)$ is a point of Y we have to show that $L(x) < f^\lambda(z_0)$ for $x < f^\lambda(y_0)$. Now to each x such that $a < x < f^\lambda(y_0)$,

there corresponds an η $(a < \eta < y_0)$, so that $f^\lambda(\eta) = x$. For such η, $L(\eta) < L(y_0) = z_0$, and so

$$L(x) = Lf^\lambda(\eta) = f^\lambda L(\eta) < f^\lambda(z_0), \quad a < x < f^\lambda(y_0),$$

as required.

For points t of T, we have

$$Lf^\lambda(t) = f^\lambda L(t) = f^\lambda(t),$$

so that $f^\lambda(t)$ is a point of either T or Z. If $f^\lambda(t)$ is a point of Z, so also, by the above argument, is $f^{-\lambda}[f^\lambda(t)] = t$, which is false. It follows that $f^\lambda(t)$ is a point of T.

We may state the result proved in the above theorem as follows:

The function f^λ maps the interval $\langle a,b\rangle$ on to itself in such a way that $\langle y,z\rangle$ intervals correspond; or, briefly, that *f^λ maps one basic interval on to another basic interval.*

Corresponding to any member δ_0 of the basic set, we have a set of intervals, which we denote by $\{\delta_\lambda\}$, derived from δ_0 by means of f^λ, which we may regard as a functional operator. It is easily seen that this set $\{\delta_\lambda\}$ is a sub-set of the basic set, not necessarily identical with it. For the removal of a set of functions from the power-set may reduce the set, Λ, of indices, and therefore the set $\{\delta_\lambda\}$ of intervals; but $L(x)$, and therefore the basic set of intervals, remain unaltered provided that the set of indices remains everywhere dense.

DEFINITION. *A 'total' power-set is a set in which $\{\delta_\lambda\}$ is identical with the basic set.* Other sets will be described as 'partial'.

The construction of a total power-set is described in § 11. I conclude this section with a theorem which is of use in subsequent work.

[7.3] *Given, in an interval $\langle a,b\rangle$, a c.n.f. function f and a step-function L, of the type described in* § 6, *then the necessary and sufficient condition that $Lf = fL$ is that f maps each basic interval of L on to another such interval.*

The necessity of the condition is clear from the proof of [7.2].

To prove the sufficiency, we consider an interval $\langle y_0, z_0\rangle$ of the basic set associated with L. By the given condition,

$$\langle f(y_0), f(z_0)\rangle \equiv \langle y_1, z_1\rangle$$

is also an interval of the basic set. For $y_0 \leqslant x \leqslant z_0$ we have $L(x) = z_0$, so that $fL(x) = z_1$; and also $y_1 \leqslant f(x) \leqslant z_1$ for such x, giving $Lf(x) = z_1$. Thus $fL = Lf$ throughout this interval, and similarly

throughout each basic interval of L. For points of T, the result follows by continuity, since t is a limit-point of Y and Z on both sides.

8. A power-set for which the set of indices is non-enumerable

We are now in a position to prove a theorem, the truth of which was conjectured in F.P. § 7.

[8.1] *If the set Λ of indices associated with a power-set P is non-enumerable, then P is a uniform set.*

For, if not, the limit function $L(x)$ has a set of basic intervals containing a sub-set $\{\delta_\lambda\}$ in one-one correspondence with the non-enumerable set Λ. But the basic intervals are non-overlapping in $\langle a, b\rangle$ and so enumerable. The contradiction implies that P is uniform.

9. The construction of rational functional powers

We now give a construction for an Nth root of a given c.n.f. function f, i.e. for a function g, such that $g^N = f$ in (a, b).

The existence of an infinity of such functions follows at once from the expression for f in canonical form, $\psi^{-1}\alpha\psi$. For this can be done in an infinity of ways, and each form gives rise to a function $g = \psi^{-1}\alpha^{1/N}\psi$, so that $g^N = f$. Our construction will exhibit the degree to which g is arbitrary.

We have $f > I$, and c.n.f. in $\langle a, b\rangle$. Take x_0 in (a, b) and write $x_1 = f(x_0)$. Choose numbers $\xi_0, \xi_1, \ldots, \xi_N$ such that

$$x_0 = \xi_0 < \xi_1 < \ldots < \xi_N = x_1.$$

Now define g in $\langle x_0, x_1\rangle$ as follows:

(i) $g(\xi_t) = \xi_{t+1} \quad (t = 0, 1, 2, \ldots, N-1)$;

(ii) g is arbitrary in each interval (ξ_t, ξ_{t+1}) $(t = 0, 1, 2, \ldots, N-2)$ consistent with being c.s.i.* in $\langle \xi_0, \xi_{N-1}\rangle$;

(iii) for $\xi_{N-1} < x \leqslant x_1$, define $g(x)$ as $f(\eta)$, where $x_0 < \eta \leqslant \xi_1$, and $g^{N-1}(\eta) = x$.

Then it is easily verified that g is c.s.i. in $\langle \xi_{N-1}, x_1\rangle$, and so, by using (ii), throughout $\langle x_0, x_1\rangle$. The definition of g is now extended to the interval $\langle a, b\rangle$ in the usual way, by defining $g(x)$ as $f^p g f^{-p}(x)$ for $x_p \leqslant x < x_{p+1}$; and, as shown in C.F. (I), § 5, g is complete and commutes with f in $\langle a, b\rangle$.

* I use *c.s.i.* for 'continuous and strictly increasing' throughout the sequel.

This completes the construction, and I now show that $g^N = f$ in (a, b).

From (i) and (iii), $g^N(\xi) = f(\xi)$ for $\xi_0 \leqslant \xi \leqslant \xi_1$. Since g is c.n.f. in $\langle a, b\rangle$, and $\xi_1 = g(\xi_0)$, we can, for each x in (a, b), find an integer p and a number ξ $(\xi_0 \leqslant \xi < \xi_1)$, so that $x = g^p(\xi)$. Hence

$$g^N(x) = g^{N+p}(\xi) = g^p f(\xi) = f g^p(\xi) = f(x) \quad (a < x < b).$$

We note that in the above construction, the sequence of increasing ξ is arbitrary in the range (x_0, x_1). In particular, $\xi_1 = g(x_0)$ is arbitrary. Thus an Nth root g of f can be constructed, so that $g(x_0) = k$, where $a < x_0 < b$ and $x_0 < k < f(x_0)$.

10. The construction of rational power-sets

I have described in § 9 the construction of an Nth root of a given function f. This construction can be applied successively to the functions $f_1, f_2, \ldots, f_n, \ldots$, where $f_n^n = f_{n-1}$, so that $f_n^{n!} = f_1$. As shown above, each member of the sequence $\{f_n\}$ is to a large degree arbitrary, but the construction can be made systematic. For example, the point x_0 in (a, b) at which the construction is commenced can be kept constant throughout the sequence; the value at x_0 of each successive member, and the sequence $\{\xi_r\}$ in each case, can be prescribed according to a fixed law. Also the completion of f_n in the fundamental sub-intervals of § 9 (ii) can be made, for example, linearly. In this way we may formulate a systematic construction for the sequence $\{f_n\}$.

By combining integer powers of the members of this sequence according to the index law [3.1], we obtain a construction for $f^{p/q}$, for all integer p, q, and thus for the power-set $P(f, \Lambda)$, where Λ is the set of all rationals. It is easily verified that the power-set so constructed satisfies the conditions of § 2.

I now show that this power-set may be made non-uniform by imposing certain conditions on the sequence $\{f_n\}$. Choose $f_2, f_3, \ldots$, so that $f_n(x_0) = k_n$, where

$$f(x_0) > k_2 > k_3 > \ldots > k > x_0.$$

This is clearly possible from the nature of the construction, and we have

$$L(x_0) = \lim_{n\to\infty} f_n(x_0) \geqslant k > x_0.$$

Thus the set is non-uniform.

If the numbers k_n are chosen so that $k_n \to x_0$ as $n \to \infty$, then the

set may be uniform or not, the point x_0 being a point of either Z or T if the set is non-uniform.

11. A non-uniform power-set with certain prescribed properties

I have proved above the existence of non-uniform power-sets; in the case discussed, the set of indices was the set of all rationals. We next consider a more general problem. The construction of power-sets contains a high degree of arbitrariness, and I demonstrate this further by the construction of a power-set for which the following are prescribed:

(i) *as base function, a function $f(x)$, c.n.f. in $\langle a,b\rangle$, and greater than I;*

(ii) *as the set of indices, an everywhere-dense corpus Λ of rationals, including* 1;

(iii) *as the limit function of the set, a step-function $L(x)$, defined in $\langle a,b\rangle$, and such that $Lf = fL$.*

Lemma. *A strictly decreasing sequence $\{1/r_n\}$ $(r_n > 0)$ can be chosen from Λ with the properties:*

(i) *r_n divides r_{n+1};*

(ii) *each member of Λ is a multiple of $1/r_n$ for some n.*

We consider the members of Λ expressed in lowest terms p/q $(q > 0)$. The set of all integers q so arising can be arranged in natural order as a sequence $\{q_n\}$, where $q_n \to \infty$ since Λ is everywhere dense. Write ρ_n for the L.C.M. of $q_1, q_2,..., q_n$. Then each member of Λ is a multiple of $1/\rho_n$ for some n, and $1/\rho_n \to 0$. Clearly $q_1 = \rho_1 = 1$.

I prove that $1/\rho_n$ belongs to Λ for all n. If $1/\rho_n$ belongs to Λ for one value of n, we have either $\rho_{n+1} = \rho_n$, in which case $1/\rho_{n+1}$ also belongs to Λ; or $\rho_{n+1} = a\rho_n = bq_{n+1}$, where a, b are co-prime integers $(a > 1)$. Now by our hypothesis s/ρ_n belongs to Λ for all integers s. Also, if p/q belongs to the corpus Λ, where p, q are co-prime, then $1/q$ also belongs to Λ, the equation $np-mq = 1$ being soluble in integers, so that

$$\frac{np}{q} = m+\frac{1}{q}.$$

Thus t/q_{n+1} belongs to Λ for all integer t. Now

$$\frac{s}{\rho_n}+\frac{t}{q_{n+1}} = \frac{as+bt}{\rho_{n+1}},$$

and, since a, b are co-prime, integers s and t can be found so that $as+bt = 1$. Therefore $1/\rho_{n+1}$ belongs to Λ. Since, in both cases, if $1/\rho_n$ is a member of Λ, so also is $1/\rho_{n+1}$, and, since $1/\rho_1 = 1$ is a member of Λ, it follows by induction that $1/\rho_n$ is a member of Λ for all values of n.

We now exclude repetitions from $\{1/\rho_n\}$, and finally have a strictly decreasing sequence $\{1/r_n\}$ as stated.

To construct the required power-set, it is sufficient to construct the sequence $\{f_n\}$, with $f_1 = f, f_n = (f_{n+1})^{N_n}$, where N_n is the integer $r_{n+1}/r_n > 1$, defined in the lemma. Then $f_n = f_1^{1/r_n}$. This sequence $\{f_n\}$ we require to converge to the given step-function $L(x)$ as limit function.

The function $L(x)$ is defined by its basic set of intervals, which we denote by $\{\delta\}$. The set $\{\delta\}$ and the set Λ of indices are ordinally similar, and so can be put into one-one correspondence, preserving order. Let λ be a member of Λ, and let δ_λ correspond to λ in such a way that f maps δ_λ on $\delta_{\lambda+1}$; this is clearly possible, the choice of intervals corresponding to λ for $0 \leqslant \lambda < 1$ being made in an arbitrary manner.

In order that each member of the set $\{f^\lambda\}$ which we propose to construct shall commute with L, the function f^λ must, by [7.3], map δ_μ on to another interval of $\{\delta\}$; and, for consistency with the above, we must have δ_μ mapped by f^λ on to $\delta_{\lambda+\mu}$, i.e. we must have

$$f^\lambda(y_\mu) = y_{\lambda+\mu}, \tag{15}$$

and

$$f^\lambda(z_\mu) = z_{\lambda+\mu}. \tag{16}$$

The values of f^λ at points of T are then determined as limits, by the continuity of f^λ.

Suppose now that the functions $f_2, f_3,\ldots, f_n$ have been constructed successively from f_1 as required. I describe the construction of f_{n+1}. Writing g for f_{n+1}, and N for N_n, we have to construct g so that $g^N = f_n$. Write ϵ for $1/r_{n+1}$, so that $1/r_n = N\epsilon$, and $g = f^\epsilon$. Take x_0, in the notation of § 9, to be y_0, and then x_1, in that notation, is $f_n(y_0) = y_{N\epsilon}$. Take $\xi_t = y_{t\epsilon}$ $(t = 0, 1,\ldots, N)$.

For (15) to be satisfied with $\lambda = \epsilon$, we must have $g(\xi_t) = \xi_{t+1}$, which is (i) in the definition of g given in § 9. The construction of g can now be completed as in § 9, the values required by (15) and (16) being consistent with this construction. We note that g is arbitrary in the open intervals (y_λ, z_λ), where $0 \leqslant \lambda < \epsilon(N-1)$.

In this way we can construct the whole sequence $\{f_n\}$. The remaining members of the power-set are then obtained as integral powers of the members of this sequence. To show that $L(x)$ is the limit function of the power-set, it is sufficient to prove that $\bar{L}(y_\mu) = z_\mu$ for all μ, where $\bar{L}$ denotes the limit function of the set. Now by (15)

$$\bar{L}(y_\mu) = \lim_{\lambda \to +0} f^\lambda(y_\mu) = \lim_{\lambda \to +0} y_{\lambda+\mu} = z_\mu,$$

as required. It follows from the properties of L that $\bar{L} \equiv L$ in $\langle a, b \rangle$, i.e. the power-set constructed has limit function L.

It is clear that the power-set is a total set, as defined in § 7.

The more general problem of constructing a power-set having prescribed properties, and for which Λ is a general enumerable set, remains to be considered.

EXISTENCE THEOREMS FOR NON-UNIFORM POWER-SETS

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1. A NUMBER of recent papers (**1, 2, 3**) have included definitions of *functional powers* f^{λ} where λ is integral, rational, or irrational. Certain *power-sets* of functions were defined; it was shown that a power-set is either *integral* or *non-integral*, and that a non-integral power-set can be further classified as *uniform* or *non-uniform*. Existence theorems for integral power-sets and for uniform, non-integral sets are almost trivial, but those for non-uniform sets are of greater interest. Such theorems have already been established for certain special sets; in the present paper we discuss the general theorems.

2. The notation of the present paper is that used in paper (**3**). The functions of a power-set P are continuous and strictly increasing (c.s.i.) in the closed interval $\langle a, b\rangle$, have nodes† at a and b, and are node-free in the open interval (a, b), except, of course, in the case of the identity function $I(x) \equiv x$. Any member of P can be chosen as base-function; denoting the chosen member by f, every other member of P can be expressed as a functional power f^{λ} of f, where the index λ is a real number uniquely defined by the function. It can be assumed without loss of generality that $f > I$ in (a, b). When the index is a positive integer n, f^{n} is the nth integral functional power of f, and, for a negative index $-n$, f^{-n} is the inverse of f^{n}; these integral powers have elementary definitions.

We write Λ for the set of indices associated with the functions of a power-set P for a chosen base-function f of P. Then Λ includes all integers and is a *corpus*, i.e. includes the sum and difference of any two members. If λ, μ are any two numbers belonging to Λ, then every member of P commutes with every other member, and $f^{\lambda}f^{\mu} = f^{\mu}f^{\lambda} = f^{\lambda+\mu}$, the products being functional. More generally, for any integers p, q,

$$(f^{\lambda})^{p}(f^{\mu})^{q} = f^{p\lambda+q\mu}.$$

† x is a *node* of a function f if $f(x) = x$.

We also have, in (a, b),

$$f^\lambda \gtreqless f^\mu \quad \text{according as} \quad \lambda \gtreqless \mu.$$

For a non-integral power-set, Λ is everywhere dense. There is then, in view of the order relation, a limit function $L(x)$,

$$L(x) = \lim_{\lambda \to +0} f^\lambda(x).$$

The power-set is said to be *uniform* or *non-uniform* according as the convergence of f^λ to L is uniform or not; a characteristic property of a uniform set is that $L(x) \equiv x$. In either case the functional equation $Lf^\lambda = f^\lambda L$ holds for all members of the power-set.

3. In the present paper we shall be concerned exclusively with non-uniform power-sets. It has been proved that there is associated with such a set a system $\{\delta\}$ of *basic intervals* δ in (a, b); these intervals do not overlap or abut, and are such that the complement T of $\{\delta\}$ in $\langle a, b\rangle$ is nowhere dense. In each closed interval $\delta = \langle y, z\rangle$, $L(x)$ is constant and equal to z, and, at each point t of T, $L(t) = t$. It has been shown (3) that a necessary and sufficient condition for the functional equation $L\phi = \phi L$ is that ϕ maps each δ-interval on another, i.e. corresponding to each δ there is a δ' such that $x \in \delta$ implies $\phi(x) \in \delta'$. Thus each member of the power-set maps each δ-interval on another. It was seen to follow from this that *the set of indices associated with a non-uniform power-set is enumerable.*

4. The main existence theorem is:

THEOREM 1. *For any enumerable everywhere-dense corpus Λ which includes unity, there is a non-uniform power-set for which Λ is the set of indices.*

In fact, this can be made more precise. In addition to Λ we can specify any suitable function f to act as base-function; to characterize the non-uniformity, we select a sub-interval $\delta = \langle y, z\rangle$, fill the open interval between δ and $\langle f(y), f(z)\rangle$ arbitrarily with an everywhere-dense system of non-overlapping and non-abutting closed intervals, and finally construct the whole system of basic intervals by mapping those already constructed with all the functions $f^p(x)$, where p is a positive or negative integer. In this way we define the intervals of constancy of a step-function $L(x)$ of the type required, and ensure that L commutes with f.

Thus, to prove Theorem 1, it is enough to prove:

THEOREM 2. *Given*

(i) *an enumerable everywhere-dense corpus* Λ *including unity,*

(ii) *a function* f *which is c.s.i. in an interval* $\langle a,b\rangle$, *has nodes at* a *and* b, *and satisfies* $f > I$ *in* (a,b), *and*

(iii) *a step-function* L *constant in each of a set of basic intervals in* (a,b) *and satisfying* $Lf = fL$,

then there is a power-set which is non-uniform, includes f, *has* Λ *for the set of indices in relation to* f *as base-function, and has* L *for its limit function.*

The basic intervals are given and form an enumerable compact system. This system is ordinally similar to the set Λ, and a one-to-one ordinal correspondence can be set up between the basic intervals and the indices in Λ, so that, if δ_λ corresponds to λ, then f^p, for any integer p, maps δ_λ on $\delta_{\lambda+p}$. To do this, select one basic interval and denote it by δ_0; write δ_1 for the interval on which f maps δ_0. The system of intervals between δ_0 and δ_1 is ordinally similar to the set of numbers between 0 and 1 belonging to Λ, and a one-to-one ordinal correspondence is set up in any way between the intervals and the indices. We now have δ_ν corresponding to $\nu \in \Lambda$ for $0 \leqslant \nu \leqslant 1$. For any other index $\lambda \in \Lambda$ we can write $\lambda = p+\nu$ where p is an integer and $0 \leqslant \nu < 1$, and we take δ_λ to be the interval on which f^p maps δ_ν.

We now construct c.s.i. functions f^λ for every $\lambda \in \Lambda$ so that f^λ maps δ_μ on $\delta_{\lambda+\mu}$. (This property already exists when λ is an integer.) We write $\delta_\lambda = (y_\lambda, z_\lambda)$, and denote the sets $\{y_\lambda\}$, $\{z_\lambda\}$ by Y, Z respectively. Then each $t \in T$ is a limit-point of Y and of Z. It is now clear that the desired functions f^λ are already known at all points of $Y+Z+T$, for we have

$$f^\lambda(y_\mu) = y_{\lambda+\mu}, \qquad f^\lambda(z_\mu) = z_{\lambda+\mu}$$

for all λ, μ belonging to Λ; for every $t \in T$, $f^\lambda(t)$ is known as a limit since f^λ is continuous and t is a limit-point of Z. These values of f^λ clearly satisfy the required relations

$$f^\lambda f^\mu = f^\mu f^\lambda = f^{\lambda+\mu},$$

$$f^\lambda \gtreqless f^\mu \quad \text{according as} \quad \lambda \gtreqless \mu$$

at all points of $Y+Z+T$. It remains, therefore, to define f^λ in each open basic interval in such a way that f^λ is c.s.i. in each closed

interval and that $f^\lambda f^\mu = f^{\lambda+\mu}$ at all points and for all indices. As long as every f^λ is c.s.i. in each closed interval, it follows that no two functions of the set can intersect, because $y_\lambda > z_\mu$ when $\lambda > \mu$; we shall therefore have the relations

$$f^\lambda \gtreqless f^\mu \quad \text{according as} \quad \lambda \gtreqless \mu$$

throughout (a, b).

In what follows it will be understood that every function f^λ is given its correct value at each point of $Y+Z+T$.

The required functions will be constructed successively. We note that when $f, f^{\lambda_1}, f^{\lambda_2}, \dots, f^{\lambda_n}$ are known, then so also is f^λ for every λ of the corpus $p_0+p_1\lambda_1+\dots+p_n\lambda_n$, the p's being arbitrary integers. In constructing a particular function, it will be found that the function is arbitrary in a number of basic intervals and is then determined in the remaining intervals partly by its chosen form in the earlier intervals and partly by the functions already constructed. If we wish to make our construction precise, we can always take a function to be linear in any interval in which it is arbitrary.

Enumerate the non-zero indices in Λ, starting with unity, as 1, $\mu_1, \mu_2, \dots$. Examine $\mu_1, \mu_2, \dots$ in turn and reject or accept each index according as it does or does not belong to the corpus formed by linear integral combinations of the indices previously accepted, taking 1 as accepted. In this way we obtain a sequence

$$S\colon\ 1, \lambda_1, \lambda_2, \dots, \lambda_n, \dots.$$

For each n, λ_n is linearly independent (as regards integral coefficients) of $1, \lambda_1, \lambda_2, \dots, \lambda_{n-1}$. Also, each member of Λ is a linear integral combination of a finite number of indices taken from S.

We shall construct $f^{\lambda_1}, f^{\lambda_2}, \dots$ successively. Suppose that $f^{\lambda_1}, f^{\lambda_2}, \dots, f^{\lambda_n}$ have been constructed, and for convenience write λ for λ_{n+1}; we proceed to construct f^λ. Write Λ_n for the corpus $p_0+p_1\lambda_1+\dots+p_n\lambda_n$, where the p's are arbitrary integers. Then our supposition is that f^α is known for every $\alpha \in \Lambda_n$, and that, if α and β belong to Λ_n, then $f^\alpha f^\beta = f^\beta f^\alpha = f^{\alpha+\beta}$. There are two cases according as λ is or is not a sub-multiple of some member of Λ_n.

Case 1. *There is no non-zero integer N such that $N\lambda \in \Lambda_n$.*

In this case it is sufficient to construct f^λ to be c.s.i., to have the required values at the points of $Y+Z+T$, and to satisfy $f^\lambda f^\alpha = f^\alpha f^\lambda$ for every $\alpha \in \Lambda_n$.

Enumerate the indices of Λ, starting with 0, and accept or reject each in turn to obtain a sequence

$$\Sigma\colon\ \nu_0,\ \nu_1,\ \nu_2,\ldots\quad (\nu_0 = 0)$$

with the property that $\nu_r - \nu_s$ does not belong to Λ_n for any r, s ($r \neq s$); thus an index is accepted unless it differs from a previously accepted index by a number belonging to Λ_n. Then any member of Λ, say μ, which does not belong to Σ is expressible uniquely in the form

$$\mu = \nu + \beta \quad \text{where} \quad \nu \in \Sigma \quad \text{and} \quad \beta \in \Lambda_n.$$

The function f^λ can now be chosen arbitrarily, subject to the usual restrictions, in the open intervals δ_0, δ_{ν_1}, δ_{ν_2},.... For any other interval δ_μ we have $\mu = \nu+\beta$, and, if $x \in \delta_\mu$, then $x = f^\beta(\eta)$ where $\eta \in \delta_\nu$. We define f^λ in δ_μ by the relation

$$f^\lambda(x) = f^\beta f^\lambda(\eta),$$

this being possible since f^β is supposed known and f^λ has already been defined in δ_ν, which contains η. This completes the construction of f^λ.

The function has the required values at the points of $Y+Z+T$ and it is easily verified that it is c.s.i. in $\langle a, b\rangle$. It only remains to verify that $f^\alpha f^\lambda = f^\lambda f^\alpha$ for every $\alpha \in \Lambda_n$. This relation is already satisfied at all points of $Y+Z+T$. Let x be any other point of (a, b), so that $x \in \delta_\mu$ for some $\mu \in \Lambda$. We can write $\mu = \nu+\beta$ where $\nu \in \Sigma$ and $\beta \in \Lambda_n$, and we have $x = f^\beta(\eta)$ for some $\eta \in \delta_\nu$. Hence

$$f^\alpha f^\lambda(x) = f^\alpha f^\beta f^\lambda(\eta) = f^{\alpha+\beta} f^\lambda(\eta) = f^\lambda(x')$$

where $$x' = f^{\alpha+\beta}(\eta) = f^\alpha f^\beta(\eta) = f^\alpha(x).$$

Thus $$f^\alpha f^\lambda(x) = f^\lambda(x') = f^\lambda f^\alpha(x)$$

as required.

Case 2. *There is a non-zero integer N such that $N\lambda \in \Lambda_n$.*

We take N to be the least positive integer having this property, and write $N\lambda = \sigma$, so that $\sigma \in \Lambda_n$. Clearly $N > 1$. We must now construct f^λ to be an Nth functional root of the known function f^σ and to satisfy $f^\alpha f^\lambda = f^\lambda f^\alpha$ for every $\alpha \in \Lambda_n$.

Denote by the symbol Λ_n' the corpus obtained by adding all integral multiples of λ to members of Λ_n. Enumerate the indices of Λ, starting with 0, and accept or reject each in turn to obtain a sequence

$$\Sigma'\colon\ \nu_0,\ \nu_1,\ \nu_2,\ldots\quad (\nu_0 = 0)$$

with the property that $\nu_r - \nu_s$ does not belong to Λ'_n for any r, s $(r \neq s)$. Then every $\mu \in \Lambda$ is expressible uniquely in the form $\mu = \nu + r\lambda + \beta$ where $\nu \in \Sigma'$, $r = 0, 1, \dots, N-1$, and $\beta \in \Lambda_n$.

The values of f^λ can now be chosen arbitrarily, subject to the usual restrictions, in the open intervals $\delta_{\nu+r\lambda}$ for each $\nu \in \Sigma'$ and $r = 0, 1, \dots, N-2$.

To determine f^λ in $\delta_{\nu+(N-1)\lambda}$, we see that, if $x \in \delta_{\nu+(N-1)\lambda}$, then numbers ξ_r $(r = 1, 2, \dots, N-1)$ can be found so that $\xi_r \in \delta_{\nu+(N-r-1)\lambda}$ and $x = f^\lambda(\xi_1)$, $\xi_1 = f^\lambda(\xi_2), \dots, \xi_{N-2} = f^\lambda(\xi_{N-1})$. Writing ξ for ξ_{N-1}, we thus see that to every $x \in \delta_{\nu+(N-1)\lambda}$ there corresponds a $\xi \in \delta_\nu$ such that $x = (f^\lambda)^{N-1}(\xi)$. We now define f^λ in $\delta_{\nu+(N-1)\lambda}$ by the relation $f^\lambda(x) = f^\sigma(\xi)$.

Thus far we have defined f^λ in all the intervals $\delta_{\nu+r\lambda}$ for every $\nu \in \Sigma'$ and $r = 0, 1, \dots, N-1$. For any other interval δ_μ we have $\mu = \nu + r\lambda + \beta$ where $\beta \in \Lambda_n$, and, if $x \in \delta_\mu$, then $x = f^\beta(\eta)$ for some $\eta \in \delta_{\nu+r\lambda}$; we define f^λ in δ_μ by the relation $f^\lambda(x) = f^\beta f^\lambda(\eta)$. This completes the definition of f^λ.

As before, the function has the required values at all the points of $Y+Z+T$ and it is easily verified that it is c.s.i. in $\langle a, b\rangle$. The proof of $f^\alpha f^\lambda = f^\lambda f^\alpha$ for every $\alpha \in \Lambda_n$ is precisely as before, and it only remains to show that $(f^\lambda)^N = f^\sigma$ throughout $\langle a, b\rangle$. This relation is already satisfied at all the points of $Y+Z+T$, so that we need only consider the open basic intervals.

From the definition of f^λ at points of $\delta_{\nu+r\lambda}$, it follows that $(f^\lambda)^N(\xi) = f^\sigma(\xi)$ for all $\xi \in \delta_\nu$. Now consider $x \in \delta_{\nu+r\lambda}$. We have $x = (f^\lambda)^r(\xi)$ for some $\xi \in \delta_\nu$, and so

$$(f^\lambda)^N(x) = (f^\lambda)^{N+r}(\xi) = (f^\lambda)^r f^\sigma(\xi).$$

Since $f^\sigma f^\lambda = f^\lambda f^\sigma$ $(\sigma \in \Lambda_n)$, we have

$$(f^\lambda)^r f^\sigma(\xi) = f^\sigma (f^\lambda)^r(\xi) = f^\sigma(x)$$

so that $(f^\lambda)^N(x) = f^\sigma(x)$, as required, in all intervals of the type $\delta_{\nu+r\lambda}$. Lastly, consider any other interval δ_μ. Then $\mu = \nu + r\lambda + \beta$ for some $\beta \in \Lambda_n$, and, if $x \in \delta_\mu$, then $x = f^\beta(\eta)$ for some $\eta \in \delta_{\nu+r\lambda}$. Hence, using the results already proved,

$$(f^\lambda)^N(x) = (f^\lambda)^N f^\beta(\eta) = f^\beta (f^\lambda)^N(\eta) = f^\beta f^\sigma(\eta) = f^\sigma f^\beta(\eta) = f^\sigma(x)$$

so that $(f^\lambda)^N = f^\sigma$ throughout $\langle a, b\rangle$.

Returning to the sequence S, we have now proved that functions

$f^{\lambda_1}, f^{\lambda_2}, \ldots$ can be constructed in this order, so that at any stage, say the nth, the functions

$$f^{p_0}(f^{\lambda_1})^{p_1}\ldots(f^{\lambda_n})^{p_n},$$

for all integers $p_0, p_1, \ldots, p_n$, constitute a power-set, the set of indices being $\Lambda_n = \{p_0+p_1\lambda_1+\ldots+p_n\lambda_n\}$. We have

$$\Lambda_0 \subset \Lambda_1 \subset \ldots \subset \Lambda_{n-1} \subset \Lambda_n \subset \Lambda$$

for all n, and every index of Λ belongs to Λ_n for some n. It follows that the whole set of functions thus constructed is a power-set having the properties described in Theorem 2; the function $L(x)$ is easily seen to be the limit function of the power-set because of the equations

$$f^{\lambda}(y_\mu) = y_{\lambda+\mu}, \qquad f^{\lambda}(z_\mu) = z_{\lambda+\mu}$$

which were used to determine the values of f^{λ} at the points of Y and Z. This completes the proof of Theorem 2.

5. The non-uniform power-set constructed above has the property that each basic interval can be mapped on any other by some function of the set, i.e., if δ and δ' are any two basic intervals, there is a function of the set which maps δ on δ'. Such a power-set is said to be *total*, because it is clear, from the fundamental properties of the basic intervals, that the power-set cannot be augmented; no new function can be added to the set so that it shall remain a power-set. A power-set which is non-uniform and not total, although its set of indices is a corpus, is called *partial*. Partial power-sets are of two kinds which will be referred to as *augmentable* and *essentially partial* respectively. An example of an augmentable set has already been provided in Theorem 2; if the construction is stopped when all the functions whose indices belong to the corpus Λ_n have been defined, we have a power-set with $L(x)$ as limit function. This set can be augmented, as we have shown, by the inclusion of further functions. An example of an essentially partial set, one which is not total and yet does not admit of the inclusion of any further functions, will be given in Theorem 4 below.

We examine the distinction more closely. Let P be a partial non-uniform power-set whose set of indices, in relation to a given base-function f, is a corpus Λ including unity. Then the limit function and the basic intervals are determinate. Let δ_0 be any chosen basic interval and write δ_λ for the interval on which f^{λ} maps δ_0 $(\lambda \in \Lambda)$. Since P is partial, $\{\delta_\lambda\}$ is not the whole system of basic intervals;

it is, however, everywhere-dense in the whole system, and a basic interval δ which does not belong to $\{\delta_\lambda\}$ corresponds to a section of Λ. Thus the interval δ corresponds to a real number τ which is not a member of Λ, and, for any $\lambda \in \Lambda$, $\lambda \gtrless \tau$ according as δ_λ is to the right or left of δ. In this way we obtain a set $\{\tau\}$ of numbers corresponding to the basic intervals not belonging to $\{\delta_\lambda\}$. Write Λ^* for the set $\Lambda+\{\tau\}$; then Λ^* may or may not be a corpus. Both cases may arise; if Λ^* is a corpus, the power-set can be augmented to have Λ^* as set of indices (Theorem 3 below), but, if Λ^* is not a corpus, the power-set is essentially partial (Theorem 4 below).

THEOREM 3. *If P is a partial non-uniform power-set whose set of indices is a corpus Λ and if the set Λ^* as defined above is a corpus, then P can be augmented to form a total power-set P^*; the set of indices of P^* is Λ^*, and the limit functions of P and P^* are the same.*

If the partial set P can be augmented to become total, then, in the notation used above, each number τ corresponding to a basic interval δ not in $\{\delta_\lambda\}$ is the index of one of the new functions. The set Λ^* includes all the indices which can arise in this way, and Λ^* is therefore the set of indices of the total power-set. We have to prove that a function f^τ, for each τ, can be constructed so that P^*, the set of all functions belonging to either P or the set $\{f^\tau\}$, is a power-set; it will follow that P^* is total. That P and P^* have the same limit function can be deduced at once from the values of the functions at the points of Y and Z.

The construction of the new functions follows closely the lines of § 4 except that the functions belonging to P are already known. The set Λ^* is already correlated with the system of basic intervals. The known functions take the right values at the points of Y and Z, and the values of the new functions at these points are determined by the usual equations

$$f^\tau(y_\mu) = y_{\mu+\tau}, \qquad f^\tau(z_\mu) = z_{\mu+\tau}$$

for all $\tau \in \Lambda^* - \Lambda$ and all $\mu \in \Lambda^*$. The values at the points of T appear as limits, as before. It remains to define the new functions in every open basic interval.

We enumerate the indices in $\Lambda^* - \Lambda$ and accept or reject each in turn to obtain a sequence

$$S^*\colon\ \lambda_1, \lambda_2, \ldots$$

such that, for every n, λ_{n+1} does not belong to the corpus

$$\Lambda+\{p_0+p_1\lambda_1+p_2\lambda_2+\ldots+p_n\lambda_n\},$$

the p's being arbitrary integers. The method of § 4 is now applied to construct functions f^{λ_1}, f^{λ_2},... successively, the only difference being that when the first n of these functions are supposed known, the corpus of indices of known functions is now

$$\Lambda_n^* = \Lambda+\{p_0+p_1\lambda_1+p_2\lambda_2+\ldots+p_n\lambda_n\}.$$

Two cases can exist as before, and the details of the construction are precisely the same as before in each case. We finally obtain a power-set P^* which includes P and has Λ^* for its set of indices; the proof of Theorem 3 is thus completed.

6. Finally we prove an existence theorem for essentially partial non-uniform power-sets.

THEOREM 4. *There are partial non-uniform power-sets which cannot be augmented to become total.*

It is sufficient to prove that there is a partial set P, as described in § 5, for which the set Λ^* as defined there is not a corpus. Let f, L, and Λ be as described in Theorem 2, and let τ be a number which does not belong to Λ and is not a sub-multiple of any member of Λ. Write Λ' for the set of numbers $\tau+\lambda$ for all $\lambda \in \Lambda$, and write $\Lambda^* = \Lambda+\Lambda'$. Then Λ^* is not a corpus; it does not, for example, contain 2τ. The set Λ^* can, however, be correlated as before with the system of basic intervals given by L so that, if $\rho \in \Lambda^*$, then f maps δ_ρ on $\delta_{\rho+1}$. We shall now prove that there is a power-set P which includes f, has Λ for the set of indices when f is taken as base-function, has L for its limit function, and is such that for every $\lambda \in \Lambda$ and $\rho \in \Lambda^*$, f^λ maps δ_ρ on $\delta_{\lambda+\rho}$ (clearly $\lambda+\rho \in \Lambda^*$). Since the system $\{\delta_\lambda\}$ for all $\lambda \in \Lambda$ is not the whole system of basic intervals, P is only partial. The remaining basic intervals are δ_θ where $\theta \in \Lambda'$, and the number associated with δ_θ in the way described in § 5 is clearly θ. Thus $\Lambda^* = \Lambda+\Lambda'$ is the set Λ^* defined in § 5. No function with index $\theta \in \Lambda'$ can be added to P since, if a function g is a member of a power-set, so, for example, is g^2; but 2θ is not a member of Λ^*. Thus P is essentially partial.

To construct the functions of P we proceed exactly as in § 4 as far as the intervals δ_λ $(\lambda \in \Lambda)$ are concerned; the existence of basic intervals δ_θ $(\theta \in \Lambda')$ does not affect this construction in any way. It

remains for us to define f^λ for each $\lambda \in \Lambda$ in every interval δ_θ, $\theta \in \Lambda'$. At the end-points of these intervals we have as before $f^\lambda(y_\theta) = y_{\theta+\lambda}$, $f^\lambda(z_\theta) = z_{\theta+\lambda}$. Let f^λ be chosen arbitrarily, subject to the usual restrictions, in δ_τ, for every $\lambda \in \Lambda$; then f^λ maps δ_τ on $\delta_{\tau+\lambda}$. For any other $\theta \in \Lambda'$ we have $\theta = \tau+\nu$ for some $\nu \in \Lambda$, and, for $x \in \delta_\theta$, we have $x = f^\nu(\xi)$ for some $\xi \in \delta_\tau$ since f^ν maps δ_τ on $\delta_{\tau+\nu}$; we define f^λ at x by $f^\lambda(x) = f^{\lambda+\nu}(\xi)$.

This completes the definition of every f^λ throughout $\langle a, b\rangle$, and the functions so obtained are easily seen to be c.s.i. and to satisfy $f^\lambda \gtreqless f^\mu$ for $\lambda \gtreqless \mu$. It remains to verify that $f^\lambda f^\mu = f^\mu f^\lambda$. The proof given in § 4 holds for all points other than those in the open intervals δ_θ, $\theta \in \Lambda'$. For $x \in \delta_\theta$, where $\theta = \tau+\nu$ $(\nu \in \Lambda)$, we have $x = f^\nu(\xi)$, $\xi \in \delta_\tau$, and by definition

$$f^\lambda f^\mu(x) = f^\lambda f^{\mu+\nu}(\xi) = f^{\lambda+\mu+\nu}(\xi) = f^{\lambda+\mu}(x).$$

Hence $f^\lambda f^\mu(x) = f^\mu f^\lambda(x)$, and the conditions for a power-set are satisfied by P. This completes the proof of the theorem.

7. To summarize the results of this and preceding papers, we have seen that there are four essentially distinct kinds of power-sets: (i) integral power-sets, (ii) uniform power-sets, whose sets of indices can, without loss of generality, be taken to be the continuum, (iii) total non-uniform power-sets (including augmentable partial sets), and (iv) essentially partial non-uniform power-sets. The existence of sets of each type has been established, and it has been proved that the set of indices of a set of type (iii) or (iv) can be any enumerable everywhere-dense corpus which includes unity.

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ON THE MINIMUM OF A BILINEAR FORM

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1. IN this paper we study the minimum of a factorizable bilinear form

$$B(x,y,z,t) = (\alpha x+\beta y)(\gamma z+\delta t), \tag{1}$$

where $\alpha, \beta, \gamma, \delta$ are real, and x, y, z, t take all integral values, subject to

$$xt-yz = \pm 1. \tag{2}$$

We suppose that $$\Delta = \alpha\delta-\beta\gamma \neq 0,$$

and we suppose also that α/β and γ/δ are irrational, so that B does not represent zero.

Two bilinear forms will be called equivalent if one can be transformed into the other by one of the two following substitutions:

$$\text{(i)} \quad \begin{cases} x = px'+qy', & y = rx'+sy', \\ z = pz'+qt', & t = rz'+st', \end{cases}$$

$$\text{(ii)} \quad \begin{cases} x = pz'+qt', & y = rz'+st', \\ z = px'+qy', & t = rx'+sy', \end{cases}$$

where p, q, r, s are integers and $ps-qr = \pm 1$. Associated with the bilinear form is the indefinite quadratic form

$$Q(x,y) = (\alpha x+\beta y)(\gamma x+\delta y),$$

and such a quadratic form has associated with it the two equivalent bilinear forms

$$(\alpha x+\beta y)(\gamma z+\delta t), \qquad (\gamma x+\delta y)(\alpha z+\beta t).$$

If two bilinear forms are equivalent, so are the corresponding quadratic forms, and conversely.

The minimum of an indefinite binary quadratic form was studied in detail by Markoff in his classical memoir.* The minimum† $M(Q)$ of $|Q|$ satisfies

$$M(Q) \leqslant \frac{1}{\sqrt{5}}|\Delta|$$

* Markoff, *Math. Annalen*, 15 (1879), 381–407; 17 (1880), 379–400. Accounts of Markoff's work are given in Bachmann, *Arithmetik der quadratischen Formen*, II, Kapitel 4, and in Dickson, *Studies in the Theory of Numbers*, chapter 7.

† We use 'minimum' to mean 'lower bound'.

for all Q, and equality occurs if and only if Q is equivalent to a multiple of

$$x^2+xy-y^2. \tag{3}$$

For all forms other than these,

$$M(Q) \leqslant \frac{1}{\sqrt{8}}|\Delta|,$$

and equality occurs if and only if Q is equivalent to a multiple of

$$x^2-2y^2. \tag{4}$$

For all forms other than these,

$$M(Q) \leqslant \sqrt{\left(\frac{25}{221}\right)}|\Delta|, \tag{5}$$

and so on. The sequence of special forms continues indefinitely, the nth form Q_n having

$$M(Q_n) = \sqrt{\left(\frac{u_n^2}{9u_n^2-4}\right)}|\Delta|,$$

where u_n takes the sequence of values

$$1, 2, 5, 13, 29, 34, \ldots,$$

the 'Markoff numbers'. For all forms not equivalent to a multiple of one of these,

$$M(Q) \leqslant \tfrac{1}{3}|\Delta|.$$

We shall prove results for the possible values of $M(B)$, the minimum of $|B|$, which, though similar to those above as far as the first two minima are concerned, show a fundamental difference when we come to the third minimum. We establish four theorems.

THEOREM 1. *For all forms B,*

$$M(B) \leqslant \frac{3-\sqrt{5}}{2\sqrt{5}}|\Delta|, \tag{6}$$

and equality occurs if and only if B is equivalent to a multiple of

$$B_1 = \left(x+\frac{1+\sqrt{5}}{2}y\right)\left(z+\frac{1-\sqrt{5}}{2}t\right), \tag{7}$$

in which case the minimum is attained.

THEOREM 2. *For all forms other than those specified in Theorem 1,*

$$M(B) \leqslant \frac{2-\sqrt{2}}{4}|\Delta|, \tag{8}$$

and equality occurs if and only if B is equivalent to a multiple of

$$B_2 = (x-\surd 2\,y)(z+\surd 2\,t), \tag{9}$$

in which case the minimum is attained.

THEOREM 3. *For all forms other than those specified in Theorems* 1 *and* 2,

$$M(B) \leqslant \frac{\surd 2-1}{3}|\Delta|, \tag{10}$$

and equality occurs if and only if B is equivalent to a multiple of

$$B_3 = (x-\surd 2\,y)\{z+(3-\surd 2)t\}, \tag{11}$$

*in which case the minimum is attained.**

THEOREM 4. *For any $\delta > 0$ there exists a set of forms, no one of which is equivalent to a multiple of another, for which*

$$M(B) > \left(\frac{\surd 2-1}{3}-\delta\right)|\Delta|,$$

and the set has the cardinal number of the continuum.

2. Throughout the paper, small Latin letters denote rational integers.

LEMMA 1. *The particular forms B_1 and B_2 have*

$$M(B_1) = \frac{3-\surd 5}{2}, \qquad M(B_2) = \surd 2-1,$$

and these minima are attained.

Proof. We have

$$\begin{aligned} B_1(x,y,z,t) &= xz-yt+\tfrac{1}{2}(1+\surd 5)yz+\tfrac{1}{2}(1-\surd 5)xt \\ &= xz-yt+yz\pm\tfrac{1}{2}(1-\surd 5), \end{aligned}$$

by (2). Hence

$$|B_1| \geqslant \min_n |n\pm\tfrac{1}{2}(1-\surd 5)| = \frac{3-\surd 5}{2},$$

since $0 < \frac{1}{2}(3-\surd 5) < \frac{1}{2}$. On the other hand

$$B_1(1,-1,0,1) = \tfrac{1}{2}(3-\surd 5).$$

Similarly, $$B_2(x,y,z,t) = xz-2yt\pm\surd 2,$$

so that $|B_2| \geqslant \surd 2-1$, and equality occurs for $x = y = z = 1$, $t = 0$.

* It may be noted here that, whereas the minima of B_1 and B_2 are attained infinitely often, that of B_3 is not.

LEMMA 2. *For the particular form B_3, we have*

$$M(B_3) = \sqrt{2}-1,$$

and the minimum is attained.

Proof. We have to show that

$$|(x-\sqrt{2}\,y)\{z+(3-\sqrt{2})t\}| \geqslant \sqrt{2}-1 \tag{12}$$

for all x, y, z, t satisfying (2). Equality obviously occurs when $x = y = z = 1$, $t = 0$.

It is clear that (12) holds when $y = 0$, and when $t = 0$; we can therefore suppose, without loss of generality, that

$$y \geqslant 1, \qquad t \geqslant 1.$$

We write (12) in the form

$$\left|\left(\frac{x}{y}-\sqrt{2}\right)\left(\frac{z}{t}+3-\sqrt{2}\right)\right| \geqslant \frac{\sqrt{2}-1}{yt}. \tag{13}$$

If $z/t \geqslant -\frac{3}{2}$, we have

$$\frac{z}{t}+3-\sqrt{2} \geqslant \left|\frac{z}{t}+\sqrt{2}\right|,$$

and (13) follows from the result proved for B_2 in the preceding lemma. A similar argument holds if $x/y \leqslant -\frac{3}{2}$, since then

$$\sqrt{2}-\frac{x}{y} \geqslant \left|3+\frac{x}{y}+\sqrt{2}\right|,$$

and
$$\left|\left(3+\frac{z}{t}-\sqrt{2}\right)\left(3+\frac{x}{y}+\sqrt{2}\right)\right| \geqslant \frac{\sqrt{2}-1}{yt},$$

by considering $B_2(z+3t, t, x+3y, y)$.

We may therefore suppose that

$$\frac{x}{y} > -\frac{3}{2} > \frac{z}{t}, \qquad \frac{x}{y}-\frac{z}{t} = \frac{1}{yt}.$$

Since
$$\frac{x}{y}+\frac{3}{2} \geqslant \frac{1}{2y}, \qquad -\frac{3}{2}-\frac{z}{t} \geqslant \frac{1}{2t}, \tag{14}$$

it follows that
$$\frac{1}{yt} \geqslant \frac{1}{2y}+\frac{1}{2t},$$

i.e. $2 \geqslant y+t$, which implies $y = t = 1$, and therefore, by the equality in (14), $x = -1$, $z = -2$. For these values,

$$|B_3| = |(-1-\sqrt{2})\{-2+(3-\sqrt{2})\}| = 1 > \sqrt{2}-1.$$

LEMMA 3. *If B is not equivalent to a multiple of B_1 or B_2, then the quadratic form Q corresponding to B satisfies*

$$M(Q) \leqslant \sqrt{\left(\frac{25}{221}\right)}|\Delta| = \frac{|\Delta|}{2{\cdot}973...}. \tag{15}$$

Proof. This follows at once from the results of Markoff which were stated in § 1.

LEMMA 4. *Suppose Q assumes a value satisfying*

$$|Q(p,q)| = \frac{|\Delta|}{\mu} \tag{16}$$

for some p, q with $(p,q) = 1$. Then the corresponding B is equivalent to a multiple of

$$(x+\beta y)\{z+(\mu+\beta)t\}, \tag{17}$$

where β satisfies

$$0 \leqslant \beta+\tfrac{1}{2}\mu \leqslant \tfrac{1}{2}. \tag{18}$$

Proof. Plainly Q is equivalent to

$$\pm\frac{\Delta}{\mu}(x+\beta' y)(x+\delta' y),$$

for some β', δ'. Hence B is equivalent to

$$\frac{\Delta}{\mu}(x+\beta' y)(z+\delta' t),$$

and by comparison of determinants,

$$\beta'-\delta' = \pm\mu.$$

Hence B is equivalent to a multiple of

$$(x+\beta' y)\{z+(\mu+\beta')t\}.$$

By the further substitution

$$x = x'+ny', \qquad y = y', \qquad z = z'+nt', \qquad t = t',$$

we can replace β' by $\beta'+n$, and can therefore ensure that

$$|\beta'+\tfrac{1}{2}\mu| \leqslant \tfrac{1}{2}.$$

If $\beta'+\frac{1}{2}\mu < 0$, we apply the substitution $x = z'$, $y = -t'$, $z = x'$, $t = -y'$ and take $\beta = -\mu-\beta'$.

LEMMA 5. *If $\frac{5}{2} < \mu \leqslant 3$, there is a unique positive number $\eta = \eta(\mu)$ such that*

$$\left(2\mu-5-\frac{2\mu}{\eta}\right)\left(2+\frac{\mu}{\eta}\right) = \frac{\mu}{\eta}. \tag{19}$$

Also $\quad \eta(3) = 3(1+\sqrt{2}),$ *and* $\eta(\mu) > \eta(3)$ *for* $\mu < 3$.

Proof. The condition (19) is the same as

$$\eta^2(2\mu-5)+\eta(\mu^2-5\mu)-\mu^2 = 0, \tag{20}$$

and this equation has obviously a unique positive root. When $\mu = 3$, it reduces to

$$\eta^2-6\eta-9 = 0,$$

and the positive root is then $3(1+\surd 2)$.

If we denote the left-hand side of (20) by $\phi(\eta,\mu)$, we have

$$\begin{aligned}\frac{\partial}{\partial\mu}\phi\{\eta(3),\mu\} &= 2\eta^2(3)+(2\mu-5)\eta(3)-2\mu \\ &\geqslant 2\{\eta^2(3)-\mu\} > 0.\end{aligned}$$

Since $\phi\{\eta(3),3\} = 0$, it follows that

$$\phi\{\eta(3),\mu\} < 0$$

for $\mu < 3$, and so $\eta(\mu) > \eta(3)$ for $\mu < 3$.

LEMMA 6. *If $\mu \geqslant 3$, there is a unique positive number $\eta = \eta(\mu)$ such that*

$$\left(\mu-1+\frac{\mu}{\eta}\right)\left(1-\frac{2\mu}{\eta}\right) = \frac{\mu}{\eta}. \tag{21}$$

Also $\quad \eta(3) = 3(1+\surd 2)$, *and* $\quad \eta > \eta(3)$ *for* $\quad \mu > 3$.

Proof. The condition (21) is the same as

$$\eta^2(\mu-1)+\eta(2\mu-2\mu^2)-2\mu^2 = 0, \tag{22}$$

and this equation has obviously a unique positive root. When $\mu = 3$, it reduces to $2\eta^2-12\eta-18 = 0$, whence $\eta(3) = 3(1+\surd 2)$.

If we denote the left-hand side of (22) by $\psi(\eta,\mu)$, we have

$$\begin{aligned}\frac{\partial}{\partial\mu}\psi\{\eta(3),\mu\} &= \eta^2(3)+(2-4\mu)\eta(3)-4\mu \\ &< \eta(3)\{\eta(3)+2-4\mu\} < 0,\end{aligned}$$

since $\eta(3) < 8 < 4\mu-2$. It follows that

$$\psi\{\eta(3),\mu\} < 0$$

for $\mu > 3$, and so $\eta > \eta(3)$.

LEMMA 7. *If $2{\cdot}96 \leqslant \mu \leqslant 3{\cdot}14$, and η has the significance given to it in Lemmas 5 and 6, then*

$$(1+\tfrac{1}{2}\mu)|3-\mu| < \frac{\mu}{\eta}, \tag{23}$$

$$|\tfrac{1}{2}\mu-\tfrac{3}{2}| < \frac{\mu}{\eta}. \tag{24}$$

Proof. Since $1+\frac{1}{2}\mu > \frac{1}{2}$, (23) implies

$$\tfrac{1}{2}|\mu-3| < \frac{\mu}{\eta},$$

which is (24).

To prove (23), we first show that $\eta < 8$. With the notation already adopted,

$$\begin{aligned}\phi(8,\mu) &= 8^2(2\mu-5)+8(\mu^2-5\mu)-\mu^2 \\ &= 7\mu^2+88\mu-320 \\ &= 7(3-\mu)^2-130(3-\mu)+7 > 0\end{aligned}$$

for $2{\cdot}96 \leqslant \mu \leqslant 3$. Similarly

$$\begin{aligned}\psi(8,\mu) &= 8^2(\mu-1)+8(2\mu-2\mu^2)-2\mu^2 \\ &= -18\mu^2+80\mu-64 \\ &= -18(\mu-3)^2-28(\mu-3)+14 > 0\end{aligned}$$

for $3 \leqslant \mu \leqslant 3{\cdot}14$. This proves $\eta < 8$.

Finally,

$$(1+\tfrac{1}{2}\mu)|3-\mu| \leqslant \left(\frac{1}{\mu}+\frac{1}{2}\right)\mu(\cdot 14) \leqslant \frac{2{\cdot}48}{2{\cdot}96}(\cdot 14)\mu < \frac{\mu}{8} < \frac{\mu}{\eta}.$$

LEMMA 8. *If $|Q(p,q)| = |\Delta|/\mu$ for some p, q with $(p,q) = 1$, then*

$$M(B) \leqslant \frac{|\Delta|}{\eta} \quad \textit{if} \quad 2{\cdot}96 \leqslant \mu \leqslant 3{\cdot}14,$$

and

$$M(B) < \frac{|\Delta|}{\eta(3)} \quad \textit{if} \quad \mu = 3,$$

unless B is equivalent to a multiple of B_3.

Proof. By Lemma 4, it will suffice to consider the form

$$(x+\beta y)\{z+(\mu+\beta)t\},$$

where β satisfies
$$0 \leqslant \beta+\tfrac{1}{2}\mu \leqslant \tfrac{1}{2}.$$

We have, by (24),
$$\beta \leqslant \tfrac{1}{2}-\tfrac{1}{2}\mu < -1+\frac{\mu}{\eta},$$

and
$$\mu+\beta \leqslant \tfrac{1}{2}+\tfrac{1}{2}\mu < 2+\frac{\mu}{\eta}.$$

If $\beta > -1-\mu/\eta$, then

$$|(1+\beta.1)\{1+(\beta+\mu).0\}| < \frac{\mu}{\eta} = \frac{|\Delta|}{\eta}.$$

If $\mu+\beta > 2-\mu/\eta$, then

$$|(1+\beta.0)\{2+(\beta+\mu)(-1)\}| < \frac{\mu}{\eta} = \frac{|\Delta|}{\eta}.$$

Hence we need only consider values of β which satisfy

$$-\tfrac{1}{2}\mu \leqslant \beta \leqslant \min\left(-1-\frac{\mu}{\eta},\ 2-\mu-\frac{\mu}{\eta}\right). \tag{25}$$

We consider the following value of B:

$$F(\beta) = (1-\beta)\{3-2(\mu+\beta)\}.$$

$F(\beta)$ is zero when $\beta = 1$ and when $\beta = \tfrac{3}{2}-\mu$, hence

$$F'(\tfrac{5}{4}-\tfrac{1}{2}\mu) = 0.$$

Since $\tfrac{5}{4}-\tfrac{1}{2}\mu > -1$, $F(\beta)$ decreases steadily for $\beta < -1$. Also

$$|F(-\tfrac{1}{2}\mu)| = (1+\tfrac{1}{2}\mu)|3-\mu| < \frac{\mu}{\eta}$$

by (23). If $\mu \leqslant 3$,

$$\left|F\left(-1-\frac{\mu}{\eta}\right)\right| = \left(2+\frac{\mu}{\eta}\right)\left(2\mu-5-\frac{2\mu}{\eta}\right) = \frac{\mu}{\eta}$$

by (19). If $\mu \geqslant 3$,

$$\left|F\left(2-\mu-\frac{\mu}{\eta}\right)\right| = \left(1-\frac{2\mu}{\eta}\right)\left(\mu-1+\frac{\mu}{\eta}\right) = \frac{\mu}{\eta}$$

by (21). As $-1-\mu/\eta < -1$, and, for $\mu \geqslant 3$, $2-\mu-\mu/\eta < -1$, the inequality

$$|F(\beta)| \leqslant \frac{\mu}{\eta}$$

is satisfied throughout the interval (25). This proves the first assertion of Lemma 8. For the second, we observe that, when $\mu = 3$,

$$|F(\beta)| < \frac{\mu}{\eta}$$

unless $\quad \beta = -1-\dfrac{\mu}{\eta} = -1-\dfrac{3}{\eta(3)} = -1-\dfrac{1}{\sqrt{2}+1} = -\sqrt{2}.$

LEMMA 9. *If* $|Q(p,q)| = |\Delta|/\mu$ *for some* p, q *with* $(p,q) = 1$, *and* $\mu \geqslant 3{\cdot}14$, *then*

$$M(B) < \frac{|\Delta|}{7{\cdot}3} < \frac{|\Delta|}{3(1+\sqrt{2})}.$$

Proof. Again it suffices to consider the form

$$(x+\beta y)\{z+(\mu+\beta)t\},$$

where $|\Delta| = \mu$. If the result were false, we should have

$$|m+\beta| \geqslant \frac{\mu}{7{\cdot}3} \quad \text{and} \quad |n+(\mu+\beta)| \geqslant \frac{\mu}{7{\cdot}3}$$

for all m and n. In particular,

$$\frac{2\mu}{7\cdot 3} \leqslant 1. \tag{26}$$

Also there would exist k and l such that

$$|\beta-k-\tfrac{1}{2}| < \frac{1}{2}-\frac{\mu}{7\cdot 3} \quad \text{and} \quad |(\mu+\beta)-l-\tfrac{1}{2}| < \frac{1}{2}-\frac{\mu}{7\cdot 3}.$$

Hence $$|\mu-(l-k)| < 1-\frac{2\mu}{7\cdot 3}.$$

If $l-k \leqslant 3$, then $$\mu < 3+1-\frac{2\mu}{7\cdot 3},$$

whence $$\mu < 4\frac{7\cdot 3}{9\cdot 3} < 3\cdot 14,$$

contrary to hypothesis. If $l-k \geqslant 4$, then

$$\mu > 4-\left(1-\frac{2\mu}{7\cdot 3}\right),$$

whence $$\mu > 3\frac{7\cdot 3}{5\cdot 3} > \frac{7\cdot 3}{2},$$

contrary to (26).

3. *Proof of Theorems* 1, 2, 3. If B is equivalent to a multiple of B_1 or B_2, then, by Lemma 1, $M(B)$ satisfies either (6) with equality or (8) with equality, and the minimum is attained. If this is not the case, then, by Lemma 3, the quadratic form Q corresponding to B satisfies (15). Thus there exist p, q with $(p,q) = 1$ such that

$$|Q(p,q)| = \frac{|\Delta|}{\mu}, \qquad \mu > 2\cdot 96.$$

If $2\cdot 96 < \mu \leqslant 3\cdot 14$, then, by Lemmas 5, 6, 8,

$$M(B) < \frac{|\Delta|}{\eta(3)} = \frac{|\Delta|}{3(1+\sqrt{2})}$$

unless $\mu = 3$ and B is equivalent to a multiple of B_3, in which case

$$M(B) = \frac{|\Delta|}{3(1+\sqrt{2})} \tag{27}$$

by Lemma 2, and the minimum is attained. Finally, (27) holds also if $\mu > 3\cdot 14$, by Lemma 9.

4. Let ω be any real number satisfying $\omega > 10$, and let

$$\Omega = (1+\sqrt{2})^{\omega}.$$

We use $\delta_1, \delta_2,\ldots$ to denote positive numbers depending only on ω which tend to zero as $\omega \to +\infty$. We shall also use the notation

$$L = O_{\omega}(M),$$

where L, M depend on ω and other variables, and $M > 0$, to mean that

$$|L| < (1+\delta_1)^{\omega} M$$

for some δ_1. If $L = O(M)$, then $L = O_{\omega}(M)$.

We define N_i for all i by

$$N_i = [i\omega] \text{ for } i \geqslant 0, \qquad N_i = -N_{-i}+2 \text{ for } i < 0. \tag{28}$$

We define a_n for all n by

$$a_{N_i} = a_{N_i-1} = 1 \text{ for all } i, \quad \text{and} \quad a_n = 2 \text{ for all other } n. \tag{29}$$

We define the numbers p_n, q_n for all n by

$$p_0 = 1, \quad q_0 = 0; \qquad p_{-1} = 0, \quad q_{-1} = 1 \tag{30}$$

and by the recurrence relations

$$p_n = a_n p_{n-1} + p_{n-2}, \qquad q_n = a_n q_{n-1} + q_{n-2}. \tag{31}$$

We put

$$\theta_n = a_{n+1} + \frac{1}{a_{n+2}+}\,\frac{1}{a_{n+3}+}\cdots \tag{32}$$

for all n, and in particular we write $\theta = \theta_0$. We shall be concerned with the bilinear form

$$B_{\theta} = (x-\theta y)\{z+(3-\theta)t\}.$$

LEMMA 10. *If $\omega \neq \omega'$, then B_{θ} is not equivalent to a multiple of $B_{\theta'}$.*

Proof. If B_{θ} is equivalent to a multiple of $B_{\theta'}$, then the corresponding irrational numbers θ, θ' are equivalent.* In this case, there exists k such that $a_n = a'_{n+k}$ for all sufficiently large n. Since, obviously,

$$\frac{1}{M}\sum_{n=1}^{M}(2-a_n) \to \frac{2}{\omega} \quad \text{as} \quad M \to \infty,$$

the corresponding numbers ω and ω' must be equal.

* Two real numbers ρ and σ are equivalent, if integers a, b, c, d can be found such that $ad-bc = \pm 1$, $\rho = (a\sigma+b)/(c\sigma+d)$.

Lemma 11. *For $n \geqslant 2$, we have*

$$a_{-n} = a_{n+1}, \tag{33}$$

$$q_{-n} = (-1)^{n-1} q_{n-1}, \tag{34}$$

$$p_{-n} + 3q_{-n} = (-1)^{n-1} p_{n-1}. \tag{35}$$

Proof. If $-n = N_i$, where $i < 0$, then $n+1 = -N_i+1 = N_{-i}-1$, by (28). Consequently, also, if $-n = N_i-1$ then $n+1 = N_{-i}$. In these cases, $a_{-n} = a_{n+1} = 1$. For all other values of n satisfying $n \geqslant 2$, $a_{-n} = a_{n+1} = 2$. This proves (33).

For (34) and (35), we note that, by (28), (29),

$$a_2 = 2, \quad a_1 = 2, \quad a_0 = 1, \quad a_{-1} = 1, \quad a_{-2} = 2.$$

Hence, by (30) and (31),

$$p_2 = 5, \quad p_1 = 2, \quad p_0 = 1, \quad p_{-1} = 0, \quad p_{-2} = 1, \quad p_{-3} = -1,$$

$$q_2 = 2, \quad q_1 = 1, \quad q_0 = 0, \quad q_{-1} = 1, \quad q_{-2} = -1, \quad q_{-3} = 2.$$

Thus (34) and (35) are valid for $n = 2$ and $n = 3$. But, if they are valid for n and $n-1$, they are also valid for $n+1$, since, on using (31) and (33),

$$q_{-n-1} = q_{-n+1} - a_{-n+1} q_{-n} = (-1)^n (q_{n-2} + a_n q_{n-1}) = (-1)^n q_n,$$

$$\begin{aligned} p_{-n-1} + 3q_{-n-1} &= (p_{-n+1} + 3q_{-n+1}) - a_{-n+1}(p_{-n} + 3q_{-n}) \\ &= (-1)^n (p_{n-2} + a_n p_{n-1}) = (-1)^n p_n. \end{aligned}$$

Lemma 12. *For all n,*

$$p_n q_{n-1} - p_{n-1} q_n = (-1)^n, \tag{36}$$

$$(p_n - \theta q_n)\theta_n = -(p_{n-1} - \theta q_{n-1}), \tag{37}$$

$$(p_n - \theta q_n)(q_{n-1} + \theta_n q_n) = (-1)^n. \tag{38}$$

Proof. Now (36) is true for $n = 0$ by (30), and follows for $n > 0$ and $n < 0$ by induction, using (31). Similarly (37) is true for $n = 0$, and follows by induction, since

$$\frac{p_{n+1} - \theta q_{n+1}}{p_n - \theta q_n} = a_{n+1} + \frac{p_{n-1} - \theta q_{n-1}}{p_n - \theta q_n}$$

and

$$\left(-\frac{1}{\theta_{n+1}}\right) = a_{n+1} + (-\theta_n).$$

Finally (38) follows, since, by (37) and (36),

$$(p_n - \theta q_n)(q_{n-1} + \theta_n q_n) = (p_n - \theta q_n) q_{n-1} - (p_{n-1} - \theta q_{n-1}) q_n = (-1)^n.$$

LEMMA 13. *For any i, if $m = N_i$ and $m' = N_{i-1}$, we have*

$$(1+\delta_2)^{-\omega} < \Omega^{-1}\left|\frac{p_{m'}-\theta q_{m'}}{p_m-\theta q_m}\right| < (1+\delta_2)^{\omega}.$$

Proof. By (37), $$\left|\frac{p_{m'}-\theta q_{m'}}{p_m-\theta q_m}\right| = \prod_{r=m'+1}^{m} \theta_r.$$

The number of factors in the product is

$$m-m' = N_i-N_{i-1} = \omega+O(1).$$

If $r \leqslant m-k$, where $k \geqslant 3$, then

$$a_{r+1} = a_{r+2} = \ldots = a_{r+k-2} = 2,$$

and so, by (32), θ_r is a continued fraction whose first $k-2$ elements are all 2. Hence, for every such r,

$$(1+\eta_k)^{-1} < \frac{\theta_r}{1+\sqrt{2}} < 1+\eta_k,$$

where $\eta_k \to 0$ as $k \to \infty$. For other values of r, we have $1 < \theta_r < 3$ by (32). We have now

$$(1+\eta_k)^{-(m-m'-k)} < \frac{\prod \theta_r}{(1+\sqrt{2})^{m-m'-k}} < (1+\eta_k)^{m-m'-k}3^k,$$

and the result follows by taking, for example, $k = [\sqrt{\omega}]$.

LEMMA 14. *For any i,*

$$\theta_{N_i} = 1+\sqrt{2}+O(\Omega^{-2}).$$

Proof. We have, on writing N for N_i,

$$\theta_N = a_{N+1}+\frac{1}{a_{N+2}+}\ldots = 2+\frac{1}{2+}\,\frac{1}{2+}\ldots,$$

where the number of 2's is $j = \omega+O(1)$. Hence

$$\theta_N = 1+\sqrt{2}+O(Q_j^{-2}),$$

where Q_j is the denominator of the jth convergent to the infinite continued fraction for $1+\sqrt{2}$. Since

$$Q_j = \frac{(1+\sqrt{2})^{j+1}-(1-\sqrt{2})^{j+1}}{2\sqrt{2}} > C\Omega,$$

where C is an absolute constant, the result follows.

LEMMA 15. *If $n = N_i$, then*

$$\frac{p_{n-1}+(3-\theta)q_{n-1}}{p_n+(3-\theta)q_n} = 2-\sqrt{2}+O(\Omega^{-2})$$

for all i.

Proof. For $i = 0$, the result reduces to

$$3-\theta = 2-\sqrt{2}+O(\Omega^{-2}),$$

which is a consequence of Lemma 14.

If $i < 0$ we have, by (34), (35), (37),

$$\frac{p_{n-1}+(3-\theta)q_{n-1}}{p_n+(3-\theta)q_n} = -\frac{p_{-n}-\theta q_{-n}}{p_{-n-1}-\theta q_{-n-1}} = \frac{1}{\theta_{-n}}.$$

Now $-n = -N_i = N_{-i}-2$. Writing $m = N_{-i}$, we have

$$\theta_{-n} = \theta_{m-2} = a_{m-1}+\frac{1}{a_m+1/\theta_m} = 1+\frac{1}{1+1/\theta_m},$$

and $$\theta_m = 1+\sqrt{2}+O(\Omega^{-2})$$

by Lemma 14. Hence the result, in this case, since

$$\left\{1+\frac{1}{1+(\sqrt{2}+1)^{-1}}\right\}^{-1} = 2-\sqrt{2}.$$

If $i > 0$, we write the fraction with which we are concerned as

$$\frac{q_{n-1}}{q_n}\left\{\frac{(p_{n-1}/q_{n-1})-\theta+3}{(p_n/q_n)-\theta+3}\right\}. \tag{39}$$

By (38), since $\theta_n > 1$,

$$\left|\frac{p_{n-1}}{q_{n-1}}-\theta\right| < \frac{1}{q_{n-1}^2}, \qquad \left|\frac{p_n}{q_n}-\theta\right| < \frac{1}{q_n^2}.$$

Now $q_n > q_{n-1} \geqslant q_{N_1-1} > q_{N_1-2} = Q_{N_1-2}$. As in the proof of the preceding lemma, $Q_{N_1-2} > C\Omega$. Hence the second factor in (39) is $1+O(\Omega^{-2})$, and it suffices to prove that

$$\frac{q_{n-1}}{q_n} = 2-\sqrt{2}+O(\Omega^{-2}). \tag{40}$$

By (31),

$$\frac{q_n}{q_{n-1}} = a_n+\frac{1}{a_{n-1}+}\,\frac{1}{a_{n-2}+}\cdots\frac{1}{a_2}$$

$$= 1+\frac{1}{1+}\,\frac{1}{2+}\,\frac{1}{2+}\cdots,$$

where the number of 2's is $\omega+O(1)$. By the argument used in the preceding lemma, this suffices to prove (40).

5. *Proof of Theorem* 4. By Lemma 10, it suffices to prove that for all x, y, z, t satisfying $xt-yz = \pm 1$,

$$|B_\theta(x,y,z,t)| > \sqrt{2}-1-\eta(\omega), \tag{41}$$

where $\eta(\omega) \to 0$ as $\omega \to +\infty$.

By Lemma 13, for given x, y, z, t we can find a value of i such that, when $m = N_i$,

$$(1+\delta_2)^{-\omega}\Omega^{-1} < \left|\frac{x-\theta y}{x+(3-\theta)y}\right| \frac{1}{(p_m-\theta q_m)^2} < (1+\delta_2)^{\omega}\Omega. \tag{42}$$

(It suffices to take the least value of i for which the first half of the inequality holds.) We make the integral unimodular substitution

$$x = p_m X+p_{m-1} Y, \qquad y = q_m X+q_{m-1} Y,$$

$$z = p_m Z+p_{m-1} T, \qquad t = q_m Z+q_{m-1} T.$$

Then

$$x-\theta y = (p_m-\theta q_m)(X-\theta_m Y),$$

$$z+(3-\theta)t = \{p_m+(3-\theta)q_m\}(Z+\theta'_m T),$$

where

$$\theta'_m = \frac{p_{m-1}+(3-\theta)q_{m-1}}{p_m+(3-\theta)q_m}.$$

By Lemmas 14 and 15,

$$\theta_m = 1+\sqrt{2}+O(\Omega^{-2}), \qquad \theta'_m = 2-\sqrt{2}+O(\Omega^{-2}). \tag{43}$$

Hence

$$B_\theta(x,y,z,t) = \Lambda(X-\theta_m Y)(Z+\theta'_m T),$$

and a comparison of determinants shows that

$$|\Lambda| = 1+O(\Omega^{-2}). \tag{44}$$

We can obviously assume that

$$|(X-\theta_m Y)(Z+\theta'_m T)| < \sqrt{2}-1, \tag{45}$$

since otherwise (41) is satisfied.

We have also

$$x+(3-\theta)y = \{p_m+(3-\theta)q_m\}(X+\theta'_m Y);$$

hence (42) can be written

$$(1+\delta_2)^{-\omega}\Omega^{-1} < \frac{1}{|\Lambda|}\left|\frac{X-\theta_m Y}{X+\theta'_m Y}\right| < (1+\delta_2)^{\omega}\Omega.$$

By (44), this can be replaced by

$$(1+\delta_3)^{-\omega}\Omega^{-1} < \left|\frac{X-\theta_m Y}{X+\theta'_m Y}\right| < (1+\delta_3)^{\omega}\Omega. \tag{46}$$

We define ν by

$$|(X-\theta_m Y)(X+\theta'_m Y)|^{\frac{1}{2}} = \nu. \tag{47}$$

Then (46) and (47) imply

$$(1+\delta_3)^{-\omega}\Omega^{-\frac{1}{2}}\nu < |X-\theta_m Y| < (1+\delta_3)^{\omega}\Omega^{\frac{1}{2}}\nu, \tag{48}$$

$$(1+\delta_3)^{-\omega}\Omega^{-\frac{1}{2}}\nu < |X+\theta'_m Y| < (1+\delta_3)^{\omega}\Omega^{\frac{1}{2}}\nu. \tag{49}$$

From (45)

$$Z+\theta'_m T = O_\omega(\Omega^{\frac{1}{2}}\nu^{-1}). \tag{50}$$

Also, since

$$(X-\theta_m Y)(Z+\theta'_m T)-(X+\theta'_m Y)(Z-\theta_m T)$$
$$= (\theta_m+\theta'_m)(XT-YZ) = O(1),$$

we have $$(X+\theta'_m Y)(Z-\theta_m T) = O(1),$$

and so $$Z-\theta_m T = O_\omega(\Omega^{\frac{1}{2}}\nu^{-1}). \tag{51}$$

By (48), (49), (50), (51),

$$X = O_\omega(\Omega^{\frac{1}{2}}\nu), \quad Y = O_\omega(\Omega^{\frac{1}{2}}\nu), \quad Z = O_\omega(\Omega^{\frac{1}{2}}\nu^{-1}), \quad T = O_\omega(\Omega^{\frac{1}{2}}\nu^{-1}),$$

and so $$XZ,\ XT,\ YZ,\ YT \quad \text{are all} \quad O_\omega(\Omega). \tag{52}$$

By (43), the coefficients of the bilinear form

$$(X-\theta_m Y)(Z+\theta'_m T)$$

differ from the corresponding coefficients in the form

$$\{X-(1+\surd 2)Y\}\{Z+(2-\surd 2)T\} = B_3(X-Y, Y, Z-T, T)$$

by amounts which are all $O(\Omega^{-2})$. Hence, by (52),

$$(X-\theta_m Y)(Z+\theta'_m T)-B_3 = O_\omega(\Omega^{-2}.\Omega) = O_\omega(\Omega^{-1}).$$

Since $|B_3| \geqslant \surd 2-1$ by Lemma 2, and

$$(1+\delta_4)^\omega(1+\surd 2)^{-\omega} \to 0 \quad \text{as} \quad \omega \to \infty,$$

this proves (41), and so proves Theorem 4.

A NOTE ON RAMANUJAN'S FUNCTION $\tau(n)$

By R. P. BAMBAH (*Lahore*) and S. CHOWLA (*Lahore*)

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1. THE function $\tau(n)$ was defined by Ramanujan as

$$\sum_{1}^{\infty} \tau(n)x^n = x \prod_{1}^{\infty} (1-x^n)^{24}$$

when $-1 < x < 1$. Ramanujan* stated and G. N. Watson proved that

THEOREM 1. $\tau(n) \equiv 0 \pmod{691}$

for almost all n.

This result was improved by A. Walfisz† to

THEOREM 2. $\tau(n) \equiv 0 \pmod{2^5.3^2.5^2.7.691}$

for almost all n.

In this note, using a method different from that of Walfisz, we improve his Theorem 2 to

THEOREM 3.

$$\tau(n) \equiv 0 \pmod{2^5.3^2.5^2.7.23.691}$$

for almost all n.

2. We first observe that the theorem,

THEOREM 4. $\tau(n) \equiv 0 \pmod{23}$

is true for almost all n,

is implicitly contained in known results, although an explicit formulation is missing in the literature. For, we have‡

$$\tau(23m+22) \equiv 0 \pmod{23}. \tag{1}$$

Again§ (for almost all n) n is divisible by an odd power of a prime of the form $23m+22$. More precisely, for almost all n, n is expressible as $p^{\alpha}n_1$, where α is odd, p is a prime of the form $23m+22$, and n_1 is prime to p. Since p^{α} is itself of this form and

$$\tau(n) = \tau(p^{\alpha})\tau(n_1),$$

* For references see G. H. Hardy, *Ramanujan* (Cambridge, 1940), 165–9. We refer to this book as *R*.

† *Travaux de l'Inst. Math. de Tbilissi*, 5 (1938), 145–52.

‡ *R*, 166.

§ *R*, 168.

then Theorem 4 follows from (1).

3. Now denote by $\sigma_k(n)$ the sum of the kth powers of the divisors of n, i.e.

$$\sigma_k(n) = \sum_{d|n} d^k.$$

Then*

$$\sigma_s(n) \equiv 0 \pmod k$$

for almost all n, if s is odd and k is any positive integer. We shall refer to this result as (A). Now

$$\tau(n) \equiv \sigma_{11}(n) \pmod{691}; \tag{2}$$

$$\tau(n) \equiv \sigma_3(n) \pmod{2^5} \quad \text{if } n \text{ is odd}; \tag{3}$$

$$\tau(n) \equiv n\sigma_9(n) \pmod{5^2}; \tag{4}$$

$$\tau(n) \equiv n\sigma_3(n) \pmod 7; \tag{5}$$

$$\tau(n) \equiv n^2\sigma_1(n) \pmod 9. \tag{6}$$

Of these results† (2) is due to G. N. Watson; (3) and (4) are due to R. P. Bambah; (5) is due to Ramanathan; (6) was obtained (independently) by Bambah and Chowla. From (3) and (10.2.11) of R,

$$\tau(4n+2) \equiv 8\sigma_3(2n+1) \pmod{2^5}, \tag{7}$$

$$\tau(4n) \equiv 0 \pmod{64}. \tag{8}$$

From (2)–(8) and (A) we obtain a new proof of Walfisz's Theorem 2. Combining Theorem 2 with Theorem 4 we obtain Theorem 3.

We wish to express our thanks to a referee for several helpful suggestions.

* *R*, 167.

† For (2) see *R*; proofs of (3), (4), (6) have been communicated for publication in *Bull. American Math. Soc.* and *J. of London Math. Soc.*; for (5) see Ramanathan, *J. of Indian Math. Soc.* 9 (1945), 55–9.

Note added 5 May 1947. Proofs of (3) and (4) have now appeared in Bambah's note in *J. of London Math. Soc.* 21 (1946), 91–3.

ON THE LIMITS OF RIEMANN APPROXIMATING SUMS

By P. HARTMAN (*Johns Hopkins*)

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LET $F(t)$ be a real-valued, bounded function on the interval $0 \leqslant t \leqslant 1$. Let $D = D(t_0, t_1,..., t_j)$ denote a division of the interval $0 \leqslant t \leqslant 1$,

$$D: \quad 0 = t_0 < t_1 < ... < t_{j-1} < t_j = 1, \tag{1}$$

and let $\Delta(D)$ denote the degree of fineness of this division,

$$\Delta(D) = \max(t_i - t_{i-1}) \quad (i = 1, 2,..., j). \tag{2}$$

Let $E = E(\xi_1, \xi_2,..., \xi_j)$ denote a set of intermediary values corresponding to the division D,

$$t_{i-1} \leqslant \xi_i \leqslant t_i \quad (i = 1, 2,..., j). \tag{3}$$

Finally, let $\sum(D, E)$ denote the Riemann approximating sum

$$\sum(D, E) = \sum_{i=1}^{j} F(\xi_i)(t_i - t_{i-1}). \tag{4}$$

A limit point of the Riemann approximating sums will be defined as the limit of a convergent sequence $\sum(D_1, E_1), \sum(D_2, E_2),...$ for which $\Delta(D_n) \to 0$ as $n \to \infty$. It is an elementary fact* that, if S denotes the set of all limit points, then S is a closed interval. Of course, S reduces to a point if $F(t)$ is Riemann-integrable; otherwise, S is the interval whose end-points are the upper and lower Darboux integrals of $F(t)$.

The object of this note is to generalize this statement to the case when $F(t)$ is a bounded k-dimensional vector function,

$$F(t) = (f_1(t), f_2(t),..., f_k(t)),$$

on the interval $0 \leqslant t \leqslant 1$. The definitions of the last paragraph are still applicable, except that $\sum(D, E)$ is now a k-dimensional vector and so are the limit points. It will be shown that *the set S of limit points is a convex set.* The dimensionality of the set may be 0, 1,..., or k, according to the function $F(t)$.

The proof of the italicized statement will depend on the notion of the convex closure of a given bounded set, which is defined as the

* This remark was made by H. Lebesgue, *Leçons sur l'Integration* (Paris, 1928), 35.

smallest closed convex set containing the given set or equivalently as the common part of all closed half-spaces containing the given set. The main part of the proof will be the following:

LEMMA. *Let $X_1, X_2,..., X_j$ and $Y_1, Y_2,..., Y_j$ denote $2j$ arbitrary k-dimensional vectors in the Euclidean $(x_1, x_2,..., x_k)$-space and let M be a constant such that*

$$|X_i| \leqslant M, \qquad |Y_i| \leqslant M \qquad (i = 1, 2,..., j). \tag{5}$$

Let S_j denote the convex closure of the set consisting of the 2^j points

$$\sum_{i=1}^{j} \tfrac{1}{2}[(X_i+Y_i)\pm(X_i-Y_i)] \tag{6}$$

obtained from the 2^j choices of the $\pm$ signs; i.e. the i-th term of the sum (6) *is either X_i or Y_i. Then every point of S_j is within a distance* kM of one of the points* (6).

The proof of the lemma will be by mathematical induction on j. The lemma is obviously true for $j = 1$ for all dimensions k, since S_1 is the line segment joining X_1 and Y_1. Suppose the lemma is true for 1, 2,..., j for all dimensions k.

The set S_{j+1} is the convex closure of the set consisting of the set S_j translated by the vector X_{j+1} and of the set S_j translated by the vector Y_{j+1}. By translating all of the sets and points involved by $-Y_{j+1}$, it may be supposed that $Y_{j+1} = 0$ and that X_{j+1} is a vector such that $|X_{j+1}| \leqslant 2M$. Now S_{j+1} is the smallest convex set containing S_j and S_j translated by the vector X_{j+1}. Actually, it is easy to see that S_{j+1} is the set consisting of all translations of S_j by the vectors tX_{j+1}, $0 \leqslant t \leqslant 1$.

Consider any point P in S_{j+1}. If P is in either S_j or S_j translated by X_{j+1}, then the statement of the lemma follows for the point P from the induction hypothesis. Thus it may be supposed that P is not in either of these sets. Since $|X_{j+1}| \leqslant 2M$, P is within a distance M of one of these two sets, say S_j; that is, starting at P and moving a distance not exceeding M along a line parallel to X_{j+1}, one encounters a point Q of S_j. But Q lies on at least one h-dimensional $(0 \leqslant h \leqslant k-1)$ polyhedron bounding or coinciding with the closed convex polyhedron S_j. Suppose that this h-dimensional polyhedron is on the hyperplane $U.X = c$ and that S_j lies in the half-space $U.X \leqslant c$, where U is a constant vector, X a variable vector, $U.X$

* The 'kM' can be improved to '$k^{\frac{1}{2}}M$'.

their scalar product, and c a (scalar) constant. It follows that there are points (6) on this hyperplane and that the ith term of (6) for such points is the vector X_i or Y_i which gives the larger scalar product $U.X_1$ or $U.Y_1$, and that, if $U.X_i$ and $U.Y_i$ are equal, either X_i or Y_i may be the ith term. By breaking (6) into two sums, according as there is or there is not a choice for the ith term, it is seen that, up to a translation by this second sum, the sub-set of S_j on the hyperplane $U.X = c$ is the convex closure of a set of points determined by sums similar to (6). It follows by the induction hypothesis that Q is within a distance $(k-1)M$ of one of the points (6). Consequently, P is within a distance kM of the same point. This completes the proof of the lemma.

To complete the proof of the italicized statement, let $\epsilon > 0$ be a fixed number and let P_1 and P_2 be any two points of the set S. Let $D_1 = D_1(t_{01}, t_{11},..., t_{j1})$ and $E_1 = E_1(\xi_{11}, \xi_{21},..., \xi_{j1})$ be a division and a corresponding set of intermediary values such that

$$|\sum (D_1, E_1) - P_1| < \epsilon. \tag{7}$$

Let $D_2 = D_2(t_{02}, t_{12},..., t_{m2})$ and $E_2 = E_2(\xi_{12}, \xi_{22},..., \xi_{m2})$ be a division and a corresponding set of intermediary values such that

$$|\sum (D_2, E_2) - P_2| < \epsilon \tag{8}$$

and such that each point t_{i1} $(i = 0, 1,..., j)$ of D_1 is also one of the points t_{i2} $(i = 0, 1,..., m)$ of D_2. This is obviously no restriction since $\Delta(D_2)$ can be chosen arbitrarily small and inserting $t_{01},..., t_{j1}$ into the division D_2 changes $\sum (D_2, E_2)$ by at most $j\Delta(D_2)C$, where C is a bound for $F(t)$,

$$|F(t)| \leqslant C \quad (0 \leqslant t \leqslant 1). \tag{9}$$

Let $h(0)$, $h(1)$,..., $h(j)$ be the integers such that

$$t_{i1} = t_{h(i),2} \quad (i = 0, 1,..., j).$$

Put
$$X_i = F(\xi_{i1})(t_{i1} - t_{i-1,1}) \quad (i = 1, 2,..., j)$$

and
$$Y_i = \sum_{l=h(i-1)+1}^{h(i)} F(\xi_{l2})(t_{l2} - t_{l-1,2}) \quad (i = 1, 2,..., j).$$

Then

$$|X_1| \leqslant \Delta(D_1)C \quad \text{and} \quad |Y_1| \leqslant \Delta(D_1)C \quad (i = 1, 2,..., j).$$

Considering the sums (6) belonging to this set of vectors X_i and Y_i, one obviously obtains $\sum (D_1, E_1)$ by choosing all $+$ signs and $\sum (D_2, E_2)$ by choosing all $-$ signs. The lemma, (7), and (8) imply that, if P is any point on the line segment joining P_1 and P_2, then

there is a sum (6) within a distance $k\Delta(D_1)C+2\epsilon$ of P. However, the corresponding sum (6) is a Riemann approximating sum $\sum(D, E)$ for which the degree of fineness $\Delta(D)$ does not exceed $\Delta(D_1)$, so that P also belongs to the set S. This implies the convexity of the set S and completes the proof.

It may be remarked that S is a *uniform* limit set in the following sense: if $\epsilon > 0$ is arbitrary, there exists a $\delta = \delta(\epsilon)$ such that, if D is any division for which $\Delta(D) < \delta$, then for any P of S, there exists a suitable set of intermediary values E satisfying

$$|\sum (D, E)-P| < \epsilon.$$

In the proof of this statement, it will be convenient to make use of the upper Darboux integral $I(f)$ of a real-valued, bounded, scalar function $f(t)$ $(0 \leqslant t \leqslant 1)$. $I(f)$ is defined as the greatest lower bound of all sums $\sum_{i=1}^{j} (t_i-t_{i-1})M_i$, where M_i is the least upper bound of $f(t)$ on the interval $t_{i-1} \leqslant t \leqslant t_i$ and $t_0, t_1,..., t_j$ is any division of $0 \leqslant t \leqslant 1$.

Let D be a fixed arbitrary division of the interval $0 \leqslant t \leqslant 1$ and let $S(D)$ denote the convex closure of the set of all points $\sum (D, E)$, for all sets of intermediary values E. It follows from the lemma, by reasoning as above, that any point of $S(D)$ is within a distance $k\Delta(D)C$ of a point $\sum (D, E)$. Consider the least value of c for which the half-space $a_1 x_1+a_2 x_2+...+a_k x_k \leqslant c$ contains the set S. It is clear that $c = I(a_1 f_1+a_2 f_2+...+a_k f_k)$, where $I(f)$ denotes the upper Darboux integral of $f(t)$. Also, from the definition of $I(f)$, it follows that, if the half-space $a_1 x_1+...+a_k x_k \leqslant c$ contains $S(D)$, then $c \geqslant I(a_1 f_1+...+a_k f_k)$. Hence $S(D)$ contains S. Consequently any point of S is within a distance $k\Delta(D)C$ of a point $\sum (D, E)$. This completes the proof.

ON A FORMULA CONNECTING ONE MEASURE OF DISTANCE WITH ANOTHER

By A. L. DIXON (*Oxford*)

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I CONSIDER two points P, Q in relation to two alternative Absolute quadrics Φ_1, Φ_2 which meet the line PQ in the points A_1, B_1 and A_2, B_2, respectively. Then, if PQ denotes the ordinary Euclidean distance from P to Q, other measures of the same distance are

$$d_1(P, Q) = \tfrac{1}{2}\log\frac{A_1 P . B_1 Q}{B_1 P . A_1 Q}$$

(with regard to Φ_1) and

$$d_2(P,Q) = \tfrac{1}{2}\log\frac{A_2 P . B_2 Q}{B_2 P . A_2 Q}$$

(with regard to Φ_2).

It is convenient to employ small letters to denote the ordinary Euclidean distances from some base-point O to other points on the line PQ, so that $A_1 P = p-a_1$, etc. Then

$$2\sinh d_1(P, Q) = \sqrt{\frac{(p-a_1)(q-b_1)}{(p-b_1)(q-a_1)}} - \sqrt{\frac{(p-b_1)(q-a_1)}{(p-a_1)(q-b_1)}}$$

$$= \frac{(a_1-b_1)(p-q)}{\sqrt{\{(p-a_1)(p-b_1)(q-a_1)(q-b_1)\}}},$$

and so $$\frac{\sinh d_1(A_2 P)\sinh d_1(B_2 Q)}{\sinh d_1(B_2 P)\sinh d_1(A_2 Q)} = \frac{(p-a_2)(q-b_2)}{(p-b_2)(q-a_2)}.$$

Thus $$d_2(P, Q) = \tfrac{1}{2}\log\frac{\sinh d_1(A_2 P)\sinh d_1(B_2 Q)}{\sinh d_1(B_2 P)\sinh d_1(A_2 Q)}$$

is the formula connecting the measures of distance with regard to the two quadrics.